Quiz 5

Name: _____

(3 + 3.5 + 3.5 = 10 points) Crucial edges in a network flow

Consider a network flow (directed) graph G with a positive integer capacity on each edge, a source s, and a sink t. We say that an edge e is *crucial* if decreasing its capacity by one will decrease the maximum flow from s to t by one.

(a) Give an example network flow graph in which at least two edges are crucial and at least three edges are not crucial. Mark each edge as crucial or not crucial.

Answer: Here is an example with three crucial edges and three edges that are not crucial. Consider the graph G with 5 vertices s, v_1, v_2, v_3, t, and 6 edges: (s, v_1), (s, v_2), (s, v_3), (v_1, t), (v_2, t), and (v_3, t) with capacities 1, 1, 1, 2, 2, and 2, respectively. The first three edges (all out of s) are crucial, while the last three edges (all into t) are not crucial.

Recall that an edge e is a *bottleneck* if increasing its capacity by one will increase the maximum flow from s to t by one. In the following two parts, indicate whether the given statement is true or false. If you claim that the statement is true, give a brief proof; otherwise, give a counterexample.

- True or False: If an edge is crucial, then it is also a bottleneck.

  Answer: False. Consider the flow network with 3 vertices s, v, and t, and two edges (s, v) and (v, t), each with capacity 1. Both the edges are crucial, but neither is a bottleneck.

- True or False: If an edge is a bottleneck, then it is also crucial.

  Answer: True.

  Let e be a bottleneck edge. Since increasing the capacity of e by increases the maximum flow, it must also increase the minimum cut. It thus follows that e must belong to *every minimum cut* separating s and t in G.

  Consider any edge e' that is in a minimum cut of G. If we decrease the capacity of e', then the value of this cut will decrease, decreasing the maximum flow in turn. Thus any edge in *any minimum cut* separating s and t in G is a crucial edge.

  The above two arguments say the following: a bottleneck edge is one that belongs to *every* minimum cut, while a crucial edge is one that belongs to *some* minimum cut. It thus follows that every bottleneck edge is also a crucial edge.