Problem Set 2 (due Wednesday, February 9)

1. (3 + 3 + 4 = 10 points) Digital signatures

Problem 1.45 parts (b), (c), and (d) of text.

2. (10 points) Optimal location of store

You are a new player in the mobile phone business and are planning to open a physical store in Manhattan, to rival Apple's well-established store there. You want to determine an optimal location to place the store. Manhattan is organized as a grid, and a simple model is to view it as an $m \times n$ grid consisting of points (i, j), $1 \le i \le m$ and $1 \le j \le n$. Each of the mn points constitutes both a potential location for the store as well as a neighborhood of homes. The distance between two points (a, b) and (c, d) in Manhattan – $D((a, b), (c, d))$ – is simply $|a - c| + |b - d|$. For example, the distance between $(5, 3)$ and $(2, 9) = |5 - 2| + |3 - 9| = 9$.

After a detailed survey, you have found out which of the mn neighborhoods are really potential customers – so you have a set C of potential customers:

$$C = \{(i, j) : 1 \le i \le m, 1 \le j \le n, \text{ neighborhood at } (i, j) \text{ is a potential customer}\}.$$

For example, the set C could be $\{(1, 4), (1, 5), (1, 8), (2, 4), (3, 5), (6, 7), (2, 9)\}$. Since you have only one store to build, you would like to locate it at a point that minimizes the total distance from all the potential customers. That is, you want to find (x, y) in the grid that minimizes

$$\sum_{(i,j) \in C} D((i, j), (x, y)).$$

Give an efficient algorithm that takes as input the $m \times n$ grid and the set C, and returns the optimal store location (x, y).

3. (10 points) Optimizing a stock sale, in hindsight

You are working for an investment firm that prides on its analysis of the past to maximize future revenues. One of the problems you repeatedly face is the following. You know the price of a particular stock for each of the last n days; let us number these days 1, 2, ..., n. Suppose during this time period, you wanted to buy 100 shares on some day and sell all the shares on some later day (within the time period). Then, which day would you have bought and which day would you have sold? For instance, suppose that n was 10, and the stock price data you had was the following.

$$10, 11, 13, 16, 12, 8, 3, 7, 10, 9.$$

Your optimal strategy would have been to buy on day 7 and sell on day 9.

Give an efficient (polynomial-time) algorithm that solves the above problem. Analyze the worst-case running time of the algorithm.

4. (3 + 4 + 3 = 10 points) Finding good gadgets using pairwise testing

You have n supposedly identical gadgets that in principle can test each other. For instance, you can connect two gadgets A and B, each of which then reports whether the other gadget is good or bad, but the answer of a bad gadget cannot be trusted. Thus, there are four possible outcomes of such a pairwise test.

Gadget A says	Gadget B says	Conclusion
B is good	A is good	both are good, or both are bad
B is good	A is bad	at least one is bad
B is bad	A is good	at least one is bad
B is bad	A is bad	at least one is bad

You are told that more than $n/2$ of the gadgets are good, and you are asked to determine all of the good gadgets.

(a) Consider the problem of finding a single good gadget. Show that $\lfloor n/2 \rfloor$ pairwise tests are sufficient to reduce the problem to one of nearly half the size.

(b) Using part (a), give a divide-and-conquer algorithm to find a single good chip using $\Theta(n)$ pairwise tests. Give and solve the recurrence that describes the number of tests.

(c) Using part (c), show that all the good chips can be identified using $\Theta(n)$ pairwise tests.