College of Computer & Information Science — Spring 2004
Northeastern University — Handout 5
CSG399: Approximation Algorithms — 25 February 2003

Problem Set 3 (discussion day: March 8)

1. PARTITION

Consider the following optimization version of the PARTITION problem. Given a set $X = \{x_1, x_2, \ldots, x_n\}$ of n integers, find a partition of X into two subsets X_1 and X_2 that minimizes the quantity:

$$\max\{\sum_{x \in X_1} x, \sum_{x \in X_2} x\}.$$

In this exercise, we derive a PTAS for the PARTITION problem using the enumeration technique we used to dervive PTASs for knapsack and bin-packing.

Consider the following heuristic for some fixed integer k: (i) Choose the k largest x_i's. (ii) Find the optimal partition of these k integers using some exhaustive method. (iii) Complete this into a partition of X by considering each of the remaining x_i's in turn and adding it to the partition which at the time has the smallest sum.

(a) (3 points) Show how to implement the above heuristic in time $O(2^k + n)$.

(b) (12 points) Prove that the above heuristic achieves an approximation ratio of $1 + \frac{1}{2+k}$.

(c) (5 points) Using the claims of parts (a) and (b), choose k appropriately such that the above heuristic yields a PTAS for PARTITION.

2. An *independent set* of an undirected graph G is a subset V' of vertices such that no two vertices in V' have an edge in G. The INDEPENDENTSET problem is to find a maximum-size independent set in G. It is known that INDEPENDENTSET is NP-complete. In this problem, we investigate the approximability of INDEPENDENTSET.

Define the *product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as $G = (V, E)$, where

$$V = \{\langle v_1, v_2 \rangle : v_1 \in V_1, v_2 \in V_2\},$$

and there is an edge between vertices $\langle u_1, u_2 \rangle$ and $\langle v_1, v_2 \rangle$ in G if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $(u_1, v_1) \in E_1$. For a positive integer n, let G^n be defined by the recurrence relation $G^{i+1} = G^i \times G$ and $G^1 = G$.

(a) (5 points) Prove that G has an independent set of size k if and only if G^n has an independent set of size k^n.

(b) (5 points) Give a polynomial-time algorithm to construct an independent set of G of size $\lceil k^{1/n} \rceil$ from any independent set of G^n of size k.

(c) **(10 points)** Using parts (a) and (b), argue that if there exists a constant c such that there is a polynomial-time c-approximation algorithm for INDEPENDENTSET, then there exists a PTAS for INDEPENDENTSET.

3. The *bottleneck TSP problem* is the problem of finding a Hamiltonian cycle in a given weighted undirected graph such that the length of the longest edge in the cycle is minimized. Assuming that the input graph satisfies the triangle inequality, show that this problem has a polynomial-time approximation algorithm with ratio 3.

4. Exercise 3.10.

5. Exercise 11.3.