College of Computer & Information Science
Northeastern University
CS7880: Network Algorithms and Analysis

Spring 2012
Handout 1
27 January 2012

Problem Set 1 (due Friday, Feb 10)

Problem 1 (10 points) λ_2 for Laplacian of $G(n, p)$

We have seen in class that the connectivity threshold in the $G(n, p)$ model is $p = \Theta((\ln n)/n)$. In particular, we had shown that when $p < (\ln n)/2n$, the graph is disconnected with high probability (i.e., with probability $1 - o(1)$), and when $p > (4 \ln n)/n$, the graph is connected with high probability. Note that much tighter bounds on the connectivity threshold are known.

Give tight (high probability) bounds on the second smallest eigenvalue of the Laplacian of $G(n, p)$ for the corresponding cases: (a) $p < (\ln n)/2n$ and (b) $p \geq (4 \ln n)/n$.

Problem 2 (10 points) Eigenvalues of the hypercube

The n-dimensional hypercube is the graph $G = (V, E)$ with $V = \{0, 1, \ldots, 2^n - 1\}$ and the set E given as follows: (i, j) is in E if the binary representations of i and j differ in exactly one bit. Find all the eigenvalues of the Laplacian of the hypercube graph.

Problem 3 (10 points) Graph properties and the Laplacian spectrum

The independence number of a graph G is the size of the largest set S of vertices such that there are no edges between any two vertices in S. The chromatic number of a graph G is the smallest number of colors needed for coloring the vertices such that no two adjacent vertices are of the same color. The diameter of the graph is the maximum pairwise distance in the graph, where the distance between any two vertices is the length of the shortest path between the vertices. Below, λ_n is the largest eigenvalue and λ_2 is the second smallest eigenvalue.

(a) Prove that the independence number of a d-regular graph is at most $N(1 - d/\lambda_n)$.

(b) Prove that the chromatic number is at least $\lambda_n/(\lambda_n - d)$.

(c) Prove that the diameter of G is at most $4d(\log n)/\lambda_2)$.

Problem 4 (10 points) Laplacian spectrum and k-partition

Let $G = (V, E)$ be an undirected graph and let $V_1, \ldots, V_k$ denote a partition of the vertex set with $|V_i| = n/k$ (assume n is divisible by k). Let e_k denote the number of edges with endpoints in two different sets of the partition. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ denote the eigenvalues of the Laplacian of G. Prove that

$$\frac{n}{k} \sum_{i=2}^{k} \lambda_i \leq 2e_k \leq \frac{n}{k} \sum_{i=n-k+2}^{n} \lambda_i.$$

Problem 5 (10 points) PageRank on the Wikipedia dataset

Write a program for computing the PageRank vector for a given directed graph, and apply it to the Wikipedia dataset (from 2009) available at

`http://haselgrove.id.au/wikipedia.htm`

You do not need to submit your solution to this problem.