These notes and those of the next few lectures on spectral graph theory are largely based on the excellent lecture notes of Jon Kelner and Dan Spielman, who have taught related courses at MIT and Yale, respectively.

- Positive semidefinite matrices
- Courant-Fischer Theorem
- Isoperimetric number of a graph
- Cheeger's inequality: lower bound on isoperimetric number

1 Positive semidefinite matrices

Definition 1. *A real matrix M is called* positive semidefinite *(PSD) if for all real vectors $x \in R^n$, we have $x^T M x \geq \mathbf{0}$.*

Lemma 1. *For any graph G, L_G is PSD.*

Proof: Using the edge union property, we obtain that $x^T L_G x = \sum_{e \in inG} x^T L_e x$. It is easy to see that if $e = (i, j)$, then $x^T L_e x$ equals $(x_i - x_j)^2$. Thus, $x^T L_G x \geq 0$, yielding the desired claim. □

Lemma 2. *M is PSD iff all of its eigenvalues are nonnegative.*

Proof: By diagonalization, we have $M = Q^T \Lambda Q$, so $x^T M x = (Qx)^T \Lambda (Qx)$. Since Q and x are both real, so is $y = Qx$. We have $y^T \Lambda y = \sum_i \lambda_i y_i^2$. If all the eigenvalues are nonnegative, then so is $\sum_i \lambda_i y_i^2$; hence M is PSD. If M is PSD, then $\sum_i \lambda_i y_i^2 \geq 0$ for all real x with $y = Qx$. By choosing appropriate y, we obtain that each $\lambda_i \geq 0$. □ The above is an alternative proof of the nonnegativity of all the eigenvalues of the Laplacian, which we showed more directly last class.

2 Courant Fischer Theorem

Theorem 1 (Courant-Fischer Theorem). *Let M be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Let S_k denote the span of $v_1, \ldots, v_k$, for $1 \leq k \leq n$, and let S_0 equal $\{0\}$. Let $S_k^\perp$ denote the orthogonal complement of S_k. Then*

$$\lambda_k = \min_{\substack{\|x\|=1 \\ x \in S_k^\perp}} x^T M x = \min_{\substack{\|x\| \neq 0 \\ x \in S_k^\perp}} \frac{x^T M x}{x^T x}.$$

Proof: Let $M = Q^T \Lambda Q$ denote the eigendecomposition of M. Then, we have

$$x^T M x = x^T Q^T \Lambda Q x = (Qx)^T \Lambda (Qx).$$

Note that M and Λ have the exact same eigenvalues. The eigenvectors of Λ are the unit vectors e_i of the standard basis. Since Q is an orthonormal matrix, we have $\|Qx\| = \|x\|$; furthermore, x is orthogonal to v_i iff Qx is orthogonal to the ith unit vector e_i of the standard basis. Thus, it suffices for us to consider the case where $M = \Lambda$.

For $M = \lambda$, we have $x^T M x = \sum_i \lambda_i x_i^2$. If we take $x \in S_{k-1}^{\perp}$, then x is orthogonal to e_i, for $1 \le i < k$; therefore, $x_i = 0$ for $1 \le i < k$. Thus, for all $k \ge 1$ we have

$$x^T M x = \sum_{i \ge k} \lambda_i x_i^2 \ge \lambda_k \sum_{i \ge k} x_i^2 = \lambda_k \|x\|^2 = \lambda_k.$$

This, together with the fact that $e_k^T M e_k = \lambda_k$ yields the desired theorem. □

Let us use the Courant-Fischer theorem to place a bound on λ_2 for the Laplacian of a ring. Recall that if M is the Laplacian of a graph $G = (V, E)$, then $x^T M x = \sum_{(i,j) \in E} (x_i - x_j)^2$. A natural unit vector x that is perpendicular to $\mathbf{1}$ and tends to keep $x^T M x$ small is to assign $x_i = 1/\sqrt{n}$ for $i = 0, 1, \ldots, n/2 - 1$ and $x_i = -1/\sqrt{n}$ for $i = n/2, \ldots, n-1$ (assuming n is even). Using this choice of x and the Courant-Fischer theorem, we obtain that

$$lambda_2 \le 2(2/\sqrt{n})^2 = 8/n.$$

Let us compare this with the actual bound which we know from our calculation last lecture. The second smallest eigenvalue is given by

$$2 - 2\cos\left(\frac{2\pi}{n}\right) = 4\sin^2\left(\frac{\pi}{n}\right) \approx \frac{4\pi^2}{n^2}$$

for n sufficiently large.

3 Graph partitioning

Definition 2. *For any two subsets $S, T \subseteq V$, let $e(S, T)$ denote the number of edges with one endpoint in S and the other endpoint in T. The* isoperimetric number *$\phi(G)$ of an undirected graph G is the following value.*

$$\min_{S \subset V, S \neq \emptyset, V} \frac{e(S, V - S)}{\min\{|S|, |V - S|\}}$$

There are a number of other closely related graph parameters. The *cut ratio* or *sparsity* of a cut $(S, V - S)$ is given by

$$\frac{e(S, V - S)}{\min\{|S|, |V - S|\}}.$$

Thus, the isoperimetric number of a graph is the cut ratio of the cut that has the smallest cut ratio.

The *conductance* of a graph is defined as the following.

$$\min_{S \subset V, S \neq \emptyset, V} \frac{e(S, V - S)}{\min\{e(S, V), e(V - S, V)\}}.$$

For a d-regular graph, it can be seen that the conductance equals $1/d$ times the isoperimetric number.

4 Cheeger's inequality: Lower bound on isoperimetric number

Cheeger's inequality gives very useful bounds on the isoperimetric number of a graph. If $\phi = \phi(G)$ is the isoperimetric number of graph G, λ_2 the second largest eigenvalue of its Laplacian, and $d_{\max}$ its maximum degree, then

$$\frac{\phi^2}{2d_{\max}} \leq \lambda_2 \leq 2\phi.$$

In this section, we will establish the "easy direction" of Cheeger's inequality. Let $C = (S, V - S)$ be any cut of G with $0 < |S| \leq n/2$. Let x denote a vector in $\{-1, 1\}^n$ characterizing this cut: $x_i = 1$ for $i \in S$ and $x_i = -1$ for $i \in V - S$. Then, we have $e(S, V - S)$, the number of edges in the cut, being given by

$$e(S, V - S) = \frac{1}{4} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

We also have

$$|S||V - S| = \frac{1}{4} \sum_{i<j} (x_i - x_j)^2.$$

Thus, we obtain

$$\min_{x\in\{-1,1\}^n} \frac{n \sum_{(i,j)\in E} (x_i - x_j)^2}{2 \sum_{i<j} (x_i - x_j)^2} \leq \phi \leq \min_{x\in\{-1,1\}^n} \frac{n \sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_{i<j} (x_i - x_j)^2}.$$

Consequently,ϕ is approximated by the following term within

$$\min_{x\in\{-1,1\}^n} \frac{\sum_{(i,j)\in E} n(x_i - x_j)^2}{\sum_{i<j} (x_i - x_j)^2}$$

to within a factor of 2. We use the above lower bound on ϕ and show that it is at least $\lambda/2$.

Theorem 2. *Cheeger: Lower bound on ϕ $\phi \geq \lambda_2/2$.*

Proof: We first relax the requirement that $x \in \{-1, 1\}^n$, then apply the Courant-Fischer formula.

$$\begin{aligned}
\phi(G) &\geq \min_{x\in\{-1,1\}^n} \frac{\sum_{(i,j)\in E} n(x_i - x_j)^2}{2 \sum_{i<j} (x_i - x_j)^2} \\
&\geq \min_{\substack{x\in \mathrm{R}^n \\ x\perp \mathbf{1}}} \frac{\sum_{(i,j)\in E} n(x_i - x_j)^2}{2 \sum_{i<j} (x_i - x_j)^2} \\
&= \min_{\substack{x\in \mathrm{R}^n \\ x\perp \mathbf{1}}} \frac{\sum_{(i,j)\in E} n(x_i - x_j)^2}{2 \sum_i x_i^2} \\
&= \frac{\lambda_2}{2}.
\end{aligned}$$

Note that since $\sum_i x_i = 0$, we have $\sum_{i<j} (x_i - x_j)^2 = \sum_i x_i^2$, an equality that is used in the third step above. □