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CS 7880: Network Algorithms and Analysis
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Linear Algebra and Graph Spectra: Overview

Lecture Outline:

- Eigenvalues, eigenvectors, and eigendecomposition
- Matrices for graphs: adjacency, Laplacian, and normalized variants
- Properties of the Laplacian

These notes and those of the next few lectures on spectral graph theory are largely based on the excellent lecture notes of Jon Kelner and Dan Spielman, who have taught related courses at MIT and Yale, respectively.

1 Eigenvalues, eigenvectors, and eigendecomposition of a matrix

Definition 1. *For an $n \times n$ matrix M, an $n \times 1$ vector v is an* **eigenvector** *with* **eigenvalue** *λ if*

$$Mv = \lambda v.$$

The proof of the following theorem can be found in most linear algebra textbooks.

Theorem 1. *Let M be a symmetric real $n \times n$ matrix. Then the following statements are true.*

1. *If v and w are eigenvectors with distinct eigenvalues, then v and w are orthogonal.*
2. *If v and w are eigenvectors with the same eigenvalue, then for any scalars a and b, $av + bw$ is an eigenvector with the same eigenvalue as v and w.*
3. *M has a full orthonormal basis of eigenvectors v_1, v_2, ..., v_n. All eigenvalues and eigenvectors are real.*
4. *M is diagonalizable. That is,*
$$M = V\Lambda V^T,$$
where V is the matrix with columns equal to the n orthonormal eigenvectors and Λ is the diagonal matrix with the ith entry in the diagonal being λ_i, the eigenvalue of the vector v_i. Thus, we have $M = \sum_i \lambda_i v_i v_i^T$. (Note that $VV^T = I$.)

2 Matrices for graphs

Fix a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$. We assume that the graph is undirected, unweighted, and has no loops. The adjacency matrix A_G had entries given by

$$A(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We define the Laplacian L of G as the following matrix.

$$L(i,j) = \begin{cases} -1 & \text{if } (i,j) \in E \\ d(i) & i = j \\ 0 & \text{otherwise} \end{cases}$$

Here $d(i)$ is the degree of i. Note that $L = D - A$, where D is the diagonal matrix with the ith diagonal entry being $d(i)$. Every $n \times n$ matrix is a linear transformation on R^n. It can be seen that L_G is the following transformation:

$$(L_G v)(i) = (d(i)(v(i) - \text{ average of } v \text{ on neighbors of } i).$$

Eigenvalues and eigenvectors of L_G. It is easy to see that the all-1s vector $\mathbf{1}$ is an eigenvector of L_G with eigenvalue 0: $L_G\mathbf{1} = \mathbf{0}$. We will see that all other eigenvalues are nonnegative. We will use the convention that the n eigenvalues of L_G are ordered in nondecreasing order: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. We will see that the first few eigenvalues tell us a lot about the graph.

3 Properties of the Laplacian

The Laplacian L_G of a graph G over n vertices can be viewed as a linear transformation over the vector space R^n: it maps each vector in R^n to another vector in R^n. We can look at the graph structure and see how a vector v gets mapped by L_G.

$$(L_G v)(i) = d_i v(i) - \sum_{(i,j) \in E} v(j).$$

Thus, the effect of L_G is replace $v(i)$ by $d_i v(i) - \sum_{(i,j)\in E} v(j)$. We use this view of L_G to establish the following basic lemma on its eigenvalues.

Lemma 1. *For any undirected graph G, the smallest eigenvalue of L_G is* 0.

Proof: It is easy to see that the all-1s vector $\mathbf{1}$ is an eigenvector with eigenvalue 0. We now show that every eigenvalue λ of L_G is nonnegative. Let v be an arbitrary eigenvector of L_G. We assume that $v(i) > 0$ for some i; otherwise, we replace v by $-v$, which is also an eigenvector with the same eigenvalue. Let i^* be the vertex that maximizes $v(i)$, $i \in V$. Then, we have:

$$\lambda v(i) = \sum_{j:(i,j) \in E} (v_i - v_j) \geq 0,$$

implying that $\lambda \geq 0$. □

Fix a vertex set $V = \{1, \ldots, n\}$. Consider a graph consisting of the singleton edge $e = (i, j)$. Then, the Laplacian L_e of this graph is given by $L_e(i,j) = L_e(j,i) = -1$, $L_e(i,i) = L_e(j,j) = 1$, and all the other entries being zero. The following properties of the Laplacian are relatively straightforward to derive.

1. **Edge union:** If G and H have the same vertex set V with disjoint edge sets, then the Laplacian of the edge union $G \cup H$ of the graphs is given by

$$L_{G \cup H} = L_G + L_H.$$

2. **Disjoint union:** If G and H are graphs over disjoint vertex sets V_G and V_H, then the Laplacian of the disjoint union of the graphs is given by

$$L_{G \sqcup H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$$

If L_G has eigenvectors $v_1, \ldots, V_n$ with respective eigenvalues $\lambda_1, \ldots, \lambda_n$, and L_H has eigenvectors $w_1, \ldots, w_m$ with respective eigenvalues $\mu_1, \ldots, \mu_m$, then $L_{G \sqcup H}$ has eigenvectors

$$v_1 \oplus 0, v_2 \oplus 0, \ldots, v_n \oplus 0, 0 \oplus w_1, \ldots 0 \ldots w_m$$

with respective eigenvalues

$$\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m.$$

3. **Product:** The product $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ may be defined as the graph with vertex set $V \times W$ and the edge set

$$\{((v_1, w), (v_2, w)) : (v_1, v_2) \in E, w \in W\} \cup \{((v, w_1), (v, w_2)) : (w_1, w_2) \in F, v \in V\}.$$

If L_G has eigenvectors $v_1, \ldots, v_n$ with respective eigenvalues $\lambda_1, \ldots, \lambda_n$ and L_H has eigenvectors $w_1, \ldots, w_m$ with respective eigenvalues $\mu_1, \ldots, \mu_m$, then $L_{G \times H}$ has, for $1 \leq i \leq n$, $1 \leq j \leq m$, eigenvector z_{ij} given by

$$z_{ij}((v, w)) = x_i(v) y_j(w)$$

with eigenvalue $\lambda_i + \mu_j$