CS7880: Network Algorithms and Analysis Tuesday, March 13 2012

Lecture Percolation Continued: Proof of Harris' Theorem

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In this lecture we continue the proof of Harris' theorem from last time. The lecture notes are supplemented by (and the figures taken from) [1].

1 Review

Recall that we are studying percolation on the 2-dimensional infinite grid G = $\mathbb{Z}^2$. Each edge in G is open with some probability, p. We care about the probability of the event E_∞ that there exists an open, infinite connected component, denoting this probability p^*. We know by the Kolmogorov 0-1 law that p^* must be either 0 or 1.

Define $\Theta(p) = \Pr[(0,0) \in$ infinite cluster of G], and note that $\Theta(p) = 0$ implies that $p^* = 0$, and that $\Theta(p) > 0$ implies $p^* = 1$. The former is true because while we are proving specifically for the origin $(0,0)$, we are actually proving for any point (i, j) in G by translational invariance.

Our goal is to prove:

Theorem 1. ***Harris' Theorem.*** *For bond percolation probability $p \leq \frac{1}{2}$ on G = $\mathbb{Z}^2$, $\Theta(p) = 0$.*

Recall that we can take any rectangle R on G as the induced subgraph formed by $n \times m$ vertices. The natural dual of this (and indeed for any planar graph) is R^h, in which we place a vertex in the face formed by four adjacent vertices in R and an edge in R^h to cross every edge in R. We can also think of this as shifting over each vertex in $R : (a, b) \rightarrow (a + \frac{1}{2}, b + \frac{1}{2})$ as visualized in Figure 1 We will be particularly concerned with two events:

- $H(R)$: the event that there is an open horizontal crossing of R
- $V(R)$: the event that there is an open *vertical* crossing of R.

If we think about constructing both a horizontal path in the primal and a vertical path in the dual, it might seem that it is impossible, because we can only use an edge in R^h when its primal edge is not open. We have our first lemma:

Lemma 2. *Exactly one of $H(R)$ and $V(R^h)$ hold.*

Proof. We did not prove this in lecture. However, for a concise proof please see [1]. □

We immediately have the following corollary:

Corollary 3. *Let R be a $k \times l - 1$ rectangle, and let R' be a $k - 1 \times l$ rectangle. Then:*

1. $\Pr_p[H(R)] + \Pr_{1-p}[V(R')] = 1$
2. *If R is an $n \times n - 1$ rectangle, then $P_{\frac{1}{2}}[H(R)] = \frac{1}{2}$*
3. *If S is an $n \times n$ square, $P_{\frac{1}{2}}[H(S)] = P_{\frac{1}{2}}[V(S)] \geq \frac{1}{2}$*

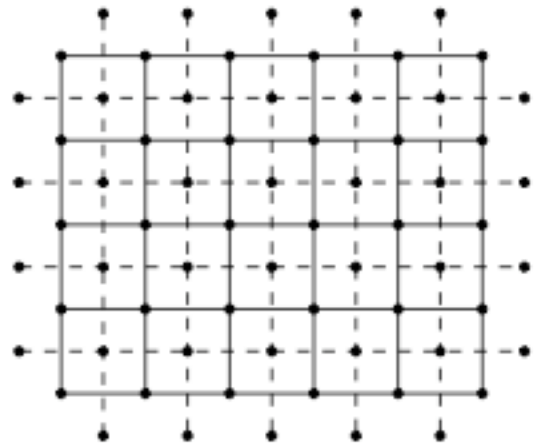

Figure 1: Portions of the lattice $L = \mathbb{Z}^2$ (solid lines) and the isomorphic dual lattice L^*

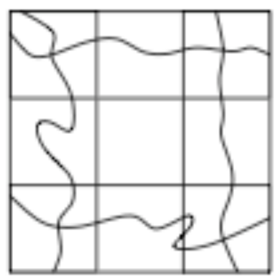

Figure 2: Four rectangles forming a square annulus

Proof. 1) Holds because R' is the dual of R, and as mutually disjoint events their independent probabilities must sum up to 1. 2) holds because if $k = l = n + 1$ then R' is R rotated 90 degrees. 2) implies 3) immediately. □

We also need to define $h_p(m, n) = \Pr[m \times n$ rectangle has an open horizontal crossing $]$, and let $h(m, n) = h_{\frac{1}{2}}(m, n)$. They key to our proof of Harris' theorem will be the following lemma:

Lemma 4. *For both the primal and dual,* $h(6n, 2n) \geq \frac{1}{2^{25}}$

2 Proof of Harris' Theorem

Assuming Lemma 4 is true for now, we directly prove Harris' theorem:

Proof. Imagine putting together 4 $6n \times 2n$ rectangles as in Figure 2 , where each rectangle overlaps with two other rectangles, creating a square annulus around a central square (not part of any of the rectangles). By Lemma 4, there is some constant probability c_1 of a path crossing one of the rectangles long-way-wise.

If a path crosses each of the rectangles, the central square is isolated in the primal (or the dual, depending on your frame of reference, but right now assume the crossing paths are occurring in the dual).

The probability of this happening is $\geq (\frac{1}{2^{25}})^4 = \frac{1}{2^{100}}$ by the FKG inequality (which we will prove later), since the probability of each crossing happening in a particular rectangle is an increasing event. In general, define A_i to be the square annulus centered on the origin with inner and outer radii 4^i and 3×4^i. The A_i are disjoint, and we can use the independence of disjoint regions to show that the origin is not surrounded by a cycle of open edges in the dual is bounded by:

$$\prod_{i=1}^{\infty} \Pr_{\frac{1}{2}}[A_i \text{ contains no open cycle surrounding } (0,0)] \leq \prod_{i=1}^{\infty}(1 - \frac{1}{2^{100}}) = 0$$

This directly implies that $\Theta(p) = 0$. □

Now we need to supply the proofs for some of the assumptions we made above. The first is the assumption that a cycle forms with an annulus with probability bounded from below by $(\frac{1}{2^{25}})^4$. To do this we need the FKG Inequality, also known as Harris' lemma, which relies on the following definition:

Definition 5. *Let X be a set with N elements, and X_p be a random subset of X obtained by selecting each $x \in X$ independently with probability p. For a family of subsets of $\mathcal{A} \subset \mathcal{P}(\mathcal{A})$, let $\Pr_p^X$ be the probability that $X_p \in \mathcal{A}$. We say $\mathcal{A}$ is **increasing** if $S \in \mathcal{A}$ and $S \subset T \subset X$ implies that $T \in \mathcal{A}$.*

For example, we can view X as a set of Boolean random variables $x_1, \ldots x_n$, and look at the event that the first five bits are one. Any event that is a superset of this, such as that the first seven bits are 1, also includes the first five bits being one, and so the original event is an increasing event. In any case:

Lemma 6. *If $\mathcal{A}$ and $\mathcal{B}$ are increasing events, then $\Pr[\mathcal{A} \bigcap \mathcal{B}] \geq \Pr[\mathcal{A}] \Pr[\mathcal{B}]$.*

Proof. Proof proceeds by induction on n. For the case that $n = 1$, the result is immediate. Now we suppose that the claim holds for $n - 1$. Let $\mathcal{A} = \mathcal{A}_0 \bigcup \mathcal{A}_1$, where $\mathcal{A}_0 = \{S \in \mathcal{A} : n \notin S\}$, $\mathcal{A}_1 = \{S - \{n\} : S \in \mathcal{A}, n \in S\}$. Also let $\mathcal{B} = \mathcal{B}_0 \bigcup \mathcal{B}_1$ be defined similarly. Note that $\mathcal{A}_0$ is an event in which $x_n = 0$ and $\mathcal{A}_1$ is an event in which $x_n = 1$. Hence $\mathcal{A}_0 \subseteq \mathcal{A}_1$ and $\mathcal{B}_0 \subseteq \mathcal{B}_1$ and $\Pr(\mathcal{A}_0) \leq \Pr(\mathcal{A}_1)$ and $\Pr(\mathcal{B}_0) \leq \Pr(\mathcal{B}_1)$ This means that:

$$[\Pr(\mathcal{A}_0) - \Pr(\mathcal{A}_1)][\Pr(\mathcal{B}_0) - \Pr(\mathcal{B}_1)] \geq 0$$

Note that:

$$\begin{aligned} \Pr(\mathcal{A}) &= (1 - p_n) \Pr(\mathcal{A}_0) + p_n \Pr(\mathcal{A}_1) \\ \Pr(\mathcal{B}) &= (1 - p_n) \Pr(\mathcal{A}_0) + p_n \Pr(\mathcal{B}_1) \end{aligned}$$

Then

$$\begin{aligned} \Pr[\mathcal{A} \bigcap \mathcal{B}] &= (1 - p_n) \Pr[\mathcal{A}_0 \bigcap \mathcal{B}_0] + p_n \Pr[\mathcal{A}_1 \bigcap \mathcal{B}_1] \\ &\geq (1 - p_n) \Pr(\mathcal{A}_0) \Pr(\mathcal{B}_0) + p_n \Pr(\mathcal{A}_1) \Pr(\mathcal{B}_1) \end{aligned}$$

where the inequality from the second line invokes the inductive hypothesis. The last thing we need to verify is the statement in the lemma, using what we have already established:

$$\begin{aligned}
\Pr(\mathcal{A})\Pr(\mathcal{B}) &= (1-p_n)^2\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_0) + p_n^2\Pr(\mathcal{B}_0)\Pr(\mathcal{B}_1) \\
&- p_n(1-p_n)\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_1) + p_n(1-p_n)\Pr(\mathcal{A}_1)\Pr(\mathcal{B}_0) \\
&\leq (1-p_n)\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_0) + p_n\Pr(\mathcal{A}_1)\Pr(\mathcal{B}_1) \\
&- p_n(1-p_n)\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_0) - p_n(1-p_n)\Pr(\mathcal{B}_0)\Pr(\mathcal{B}_1) \\
&+ p_n(1-p_n)\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_1) - p_n(1-p_n)\Pr(\mathcal{A}_1)\Pr(\mathcal{B}_0) \\
&= (1-p_n)\Pr(\mathcal{A}_0)\Pr(\mathcal{B}_0) + p_n\Pr(\mathcal{A}_1)\Pr(\mathcal{B}_1) - p_n(1-p_n)[\Pr(\mathcal{A}_0)-\Pr(\mathcal{A}_1)][\Pr(\mathcal{B}_0)-\Pr(\mathcal{B}_1)] \\
&\leq \Pr[\mathcal{A}\bigcap\mathcal{B}]
\end{aligned}$$

□

Next we need the following lemma:

Lemma 7. *Let R be a $m \times 2n$ rectangle, for $m \geq n$, and let S be an $n \times n$ square in R in the lower left corner. Define $X(R)$ as the event that there is a vertical open crossing of S, call it P, as well as a horizontal open crossing from some point in P to the right edge of S, as in Figure 3. Then*

$$\Pr[X(R)] \geq \Pr_{\frac{1}{2}}(V(S))\Pr_{\frac{1}{2}}(H(R))/2$$

Proof. Suppose that $V(S)$ holds. Then let the left-most path P that vertically crosses S, LV(S), be some path P_1. Note that all edges to the right of P_1 are independent of it. Suppose that we let P be the path formed by both P_1 and its reflection across the top of S (though the reflection is not necessarily open). There is some probability $\Pr_{\frac{1}{2}}(H(R))$ there is a horizontal path across R, and with probability $\geq \Pr_{\frac{1}{2}}(H(R))/2$ it crosses P in the region of S. In addition since $V(S)$ is an increasing event of $LV(S)$, and $LV(S)$ is independent of all edges to the right of it, we have $\Pr_{\frac{1}{2}}(X(R)|LV(S)) = \Pr_{\frac{1}{2}}(X(R))\Pr_{\frac{1}{2}}(LV(S)) \geq \Pr_{\frac{1}{2}}(V(S))\Pr_{\frac{1}{2}}(H(R))/2$. □

Finally, we are ready to conclude our proof of Lemma 4.

Proof. Consider two rectangles, $R = [0, 2n] \times [0, 2n]$ and $R' = [n, 3n] \times [0, 2n]$, and their intersection, part of which we will label the square S of size $[0, n] \times [0, n]$, as in Figure 4. If $X(R)$, $X(R')$, and $H(S)$ all hold, then so does $H(R \bigcup R')$, since any horizontal crossing of S meets any vertical crossing. Thus by Lemma 6, Lemma 7, and Corollary 3.2, we have:

$$\Pr(H(R \bigcup R')) \geq \Pr(H(R))^2 \Pr(V(S))^2 \Pr(H(S))/2 = (\frac{1}{2})^7$$

By applying this to rectangles R of size $2\rho n \times 2n$, for $\rho \geq 1$ we can get the desired constant. □

References

[1] Béla Bollobás and Oliver Riordan. A short proof of the harris-keston theorem. *Bull. London Math. Soc.*, 38:470–484, August 2006.

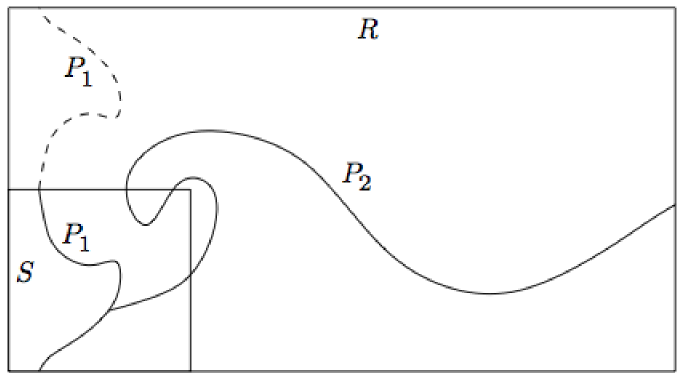

Figure 3: A rectangle R and square S inside it, drawn with paths (solid curves) whose presence as open paths would imply $X(R)$

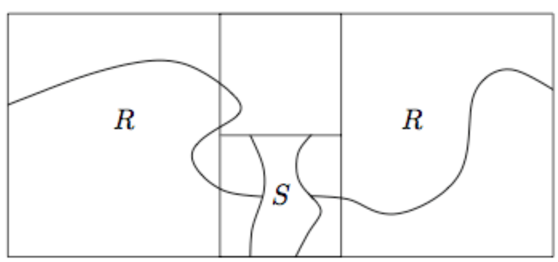

Figure 4: The overlapping rectangles R and R' with the square S in their intersection. The paths drawn show that $X(R)$ holds, as well as the reflected equivalent for R'. If $H(S)$ also holds, then so does $H(R \bigcup R')$