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Percolation on $\mathbb{Z}^2$

The following notes are largely based on the excellent coverage of percolation by Bollobas and Riordan [BO06].

1 Critical Probability of Percolation on $\mathbb{Z}^2$

In the last lecture, we showed that the critical probability p_c of percolation on $\mathbb{Z}^2$ satisfies $p_c \geq \frac{1}{3}$. This time we will prove an upper bound of p_c such that $p_c \leq \frac{2}{3}$. First, we need to give out some definitions we will work with.

Definition 1. *Let $\theta(p)$ denote the probability that there is an infinite connected component in $\mathbb{Z}^2$ with percolation probability p".*

Definition 2. *Let $\theta_x(p)$ denote the probability that there is an infinite connected component in $\mathbb{Z}^2$ containing the point $x \in \mathbb{Z}^2$ with percolation probability p. Specially, $\theta_0(p)$ denote the probability that there is an infinite connected component in $\mathbb{Z}^2$ containing the origin with percolation probability p.*

To prove the above upper bound, we first show that $\theta_0(p) > 0$, $(p > \frac{2}{3})$. By the fact that $\theta(p) \geq \theta_0(p)$, we can say that $\theta(p) > 0$, $(p > \frac{2}{3})$. Then, by Kolmogrov 0-1 Law, we can claim that $\theta(p) = 1$, $(p > \frac{2}{3})$. Thus, $\frac{2}{3}$ is an upper bound for the critical probability p_c of percolation on $\mathbb{Z}^2$.

Now, we prove that if $p > \frac{2}{3}$, then $\theta_0(p) > 0$.

First, we construct $\mathbb{Z}^{2*}$ the dual of $\mathbb{Z}^2$ by putting a vertex in every face and placing an edge between two vertices corresponding to two faces which share an edge. Each edge in $\mathbb{Z}^{2*}$ will intersect with a corresponding edge in $\mathbb{Z}^2$. As we can see, each edge in $\mathbb{Z}^{2*}$ has a one-to-one mapping with an edge in $\mathbb{Z}^2$. Furthermore, we can say that each edge in $\mathbb{Z}^{2*}$ is open (or selected) if and only if that corresponding edge in $\mathbb{Z}^2$ is closed (or not selected).

Second, we introduce a special subgraph of $\mathbb{Z}^2$, $B(m)$ a square box with side length $2m$ containing the vertices in $\{-m, \ldots, 0, \ldots, m\} \times \{-m, \ldots, 0, \ldots, m\}$. Now, we define two events G_m and F_m as following.

Definition 3. *G_m: Event that every edge in $B(m)$ is open.*

Definition 4. *F_m: Event that there is an open circuit in $\mathbb{Z}^{2*}$ containing $B(m)$ in its interior.*

Now, we let $M(n)$ denote the number of circuits of length n surrounding the origin. Since the number of paths of length n is no greater than $4 \cdot 3^{n-1}$, $M(n)$ satisfies,

$$\begin{aligned} M(n) &\leq \# \text{ of starting points } \times 4 \cdot 3^{n-1} \\ &\leq \text{Poly}(n) \cdot 4 \cdot 3^{n-1}. \end{aligned} \tag{1}$$

Then, the probability that there exists a circuit of length n around the origin that is open in $\mathbb{Z}^{2*}$. is no greater than $\text{Poly}(n) \cdot 4 \cdot 3^{n-1}(1-p)^n \leq \text{Poly}(n) \cdot 4 \cdot (3(1-p))^{n-1}$.

As we can see, $G_m \bigwedge \overline{F}_m \Rightarrow$; there is a shortest path of length $\geq m$ from the origin to the outside of the box $B(m)$. So, the probability that there is a shortest path of length $\geq m$ from the origin to the outside of the box $B(m)$ is greater than $\Pr(G_m) \cdot \Pr(\overline{F}_m)$.

$$\Pr(G_m) \cdot \Pr(\overline{F}_m) \geq \Pr(G_m) \cdot [1 - \sum_{n \geq 8m}^{\infty} 4 \cdot \text{Poly}(n)(3(1-p))^{n-1}]. \tag{2}$$

By Eq. 2, when $p > \frac{2}{3}$, for sufficiently large m, we will have,

$$\sum_{n \geq 8m}^{\infty} 4 \cdot \text{Poly}(n)(3(1-p))^{n-1} < 1. \tag{3}$$

Meanwhile, $\Pr(G_m) = (p)^{4m^2} > 0$. Then, we can claim when $m \to \infty$, the probability that there is a shortest path of length $\geq m$ from the origin to the outside of the box $B(m)$ is greater than zero. By the fact that this event implies that there is an infinite connected component in $\mathbb{Z}^2$. Thus, the probability that there is an infinite connected component in $\mathbb{Z}^2$. is greater than zero. By Kolmogrov 0-1 Law, we can say that when $p > \frac{2}{3}$, the probability that there is an infinite connected component in $\mathbb{Z}^2$ equals one. Hence, $\frac{2}{3}$ is an upper bound for the critical probability of percolation in $\mathbb{Z}^2$.

2 Harris's Theorem: $\theta(\frac{1}{2}) = 0$

Definition 5. *Let $H(R)$ denote the event that there exists an open path crossing R horizontally. Similarly, let $V(R)$ denote the event that there exists an open path crossing R vertically.*

Definition 6. *Let R^h denote the horizontal dual of R. Similarly, let R^v denote the vertical dual of R.*

Lemma 1. *Exactly one of the events of $H(R)$ and $V(R^h)$ holds.*

Corollary 1. *Given R is a $k \times (l-1)$ rectangle, R' is a $(k-1) \times l$ rectangle, then we will have the following hold.*

i) $P_p\{H(R)\}+P_{1-p}\{V(R')\}=1$;

ii) If R is an $(n+1) \times n$ rectangle, then $P_{\frac{1}{2}}\{H(R)\} = \frac{1}{2}$;

iii) If S is $n \times n$ square, then $P_{\frac{1}{2}}\{H(S)\} = P_{\frac{1}{2}}\{V(S)\} \geq \frac{1}{2}$.

Proof:

i) By Lemma 1, we will have $\mathrm{P}_p\{H(R)\} + \mathrm{P}_p\{V(R^h)\} = 1$. Observe that R^h in $\mathbb{Z}^{2*}$ is isomorphic to R' in $\mathbb{Z}^2$, and by the fact that the edge in $\mathbb{Z}^{2*}$ is open if and only if the corresponding edge in $\mathbb{Z}^2$ is closed, we will have $\mathrm{P}_p\{V(R^h)\} = \mathrm{P}_{1-p}\{V(R')\}$ hold. Hence, $\mathrm{P}_p\{H(R)\} + \mathrm{P}_{1-p}\{V(R')\} = 1$.

ii) By i), this can be immediately proved by setting $p = \frac{1}{2}$, $k = l = n+1$.

iii) By ii), given a $(n+1) \times n$ rectangle R, we will have $\mathrm{P}_{\frac{1}{2}}\{H(R)\} = \frac{1}{2}$. Suppose S is a $n \times n$ square, we can see that $H(R)$ implies $H(S)$. Hence, we will have $\mathrm{P}_{\frac{1}{2}} H(S) \geq \mathrm{P}_{\frac{1}{2}} H(R) = \frac{1}{2}$. Similarly, we can get $\mathrm{P}_{\frac{1}{2}}\{V(S)\} \geq \frac{1}{2}$.

□

Definition 7. *Let $h_p(m,n)$ denote the probability that there exists a horizontal open crossing in a $m \times n$ rectangle with percolation probability p; Specifically, we let $h(m,n) = h_{\frac{1}{2}}(m,n)$.*

Lemma 2. $h(6n, 2n) \geq 2^{-25}$.

Proof: We will cover this proof in the next lecture. □

Theorem 1 (Harris's Theorem). $\theta(1/2) = 0$.

Proof: First, construct four overlapping $6n$ by $2n$ rectangles in $\mathbb{Z}^{2*}$ in the way in Fig. 1 (taken from [BO06].

By Lemma 2, the probability for an open crossing in each $6n$ by $2n$ rectangle is at least 2^{-25}. By FKG inequality (which will be covered in the next lecture), the probability for all four crossing in these four $6n$ by $2n$ rectangles is at least $\gamma = 2^{-100}$.

For $k \geq 1$, let A_k be the small square surrounded by those four $6n$ by $2n$ rectangles in $\mathbb{Z}^{2*}$ which is centered on $(1/2, 1/2)$. Meanwhile, the small centered square and the outer big square have radii 3^k and 3^{k+1}, respectively.

Now, let E_k be the event that A_k contains an open dual cycle surrounding the interior of A_k. Then $\mathrm{P}(E_k) \geq \gamma$. Since those crossings surrounding A_k varying by k are disjoint, the events E_k are independent. If E_k holds, then no

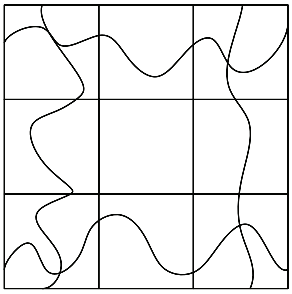

Figure 1: Four $6n$ by $2n$ rectangles overlapping

point inside A_k in $\mathbb{Z}^2$ can be reached from any point in $\mathbb{Z}^2$ outside the cycle formed by those four open crossing in $\mathbb{Z}^{2*}$. Thus, r_0 the radius of A_k is bounded by 3^{k+1}. Then, we will have

$$\mathrm{P}\{r_0 \geq 3^{l+1}\} \leq \mathrm{P}\{\bigcap_{k=1}^{l} \overline{E}_k\} = \prod_{k=1}^{l} \mathrm{P}(\overline{E}_k) \leq (1-\gamma)^l. \tag{4}$$

By Eq. 4, we have

$$\mathrm{P}\{r_0 = \infty\} \leq \mathrm{P}\{r_0 \geq n\} \leq (1-\gamma)^{\log_3 n - 1} = n^{\log_3(1-\gamma)}/(1-\gamma) \leq n^{-c}. \tag{5}$$

where, c is an constant for the above inequality's holding.

Now, we can see that $\theta_0(\frac{1}{2}) = 0$. Since it is an infinite lattice, by symmetry, for any vertex $x = (i,j) \in \mathbb{Z}^2$, we will have $\theta_x(1/2) = \theta_0(1/2) = 0$. And, there are countable infinite vertices in $\mathbb{Z}^2$. Thus, $\theta(1/2)$ satisfies following,

$$\theta(1/2) = \sum_{\forall x=(i,j)\in\mathbb{Z}^2} \theta_x(1/2) = 0. \tag{6}$$

□

References

[BO06] B. Bollobás and Riordan O. *Percolation.* Cambridge University Press, 2006.