College of Computer & Information Science — Fall 2014
Northeastern University — Handout 18
CS7800: Advanced Algorithms — 11 November 2014

Sample Solution to Problem Set 5

1. (10 points) Single-variable linear program

Exercise 29.5-9 of text.

Answer: The dual D is to minimize sy subject to $ry \geq t$ and $y \geq 0$.

1. P and D are feasible and bounded: $r > 0$ and $s \geq 0$, or $r < 0$ and $t \leq 0$, or $r = 0$ and $s = 0$ and $t = 0$.
2. P feasible but D infeasible: $r < 0$ and $t > 0$, or $r = 0$, $s \geq 0$ and $t > 0$.
3. P infeasible but D feasible: $r > 0$ and $s < 0$, or $r = 0$, $t \leq 0$ and $s < 0$.
4. P and D both infeasible: $r = 0$, $s < 0$, $t > 0$.

2. (10 points) LP for single-source shortest paths

Exercise 29.2-3 of text.

Answer: Here is one linear program for the problem.

$$\begin{aligned} \max \sum_v d_v & & \\ d_u + w(u,v) - d_v &\geq 0 \quad \text{for each edge } (u,v) \\ d_s &= 0 \\ d_v &\geq 0 \quad \text{for each vertex } v \end{aligned}$$

We claim that an optimal solution to the LP, given by $\{d_v^* : v \in V\}$ yields the shortest path distances. Let δ_v denote the shortest path weight from s to v for each v in V. Since δ_v's are nonnegative and satisfy triangle inequality, and $\delta_s = 0$, $d_v = \delta_v, \forall v$ is a feasible solution to the LP.

We now argue that δ is the optimal solution for the LP. That is, $\delta_v = d_v^*$ for all v. Suppose not, for the sake of contradiction; i.e., $\sum_v d_v^* > \sum_v \delta_v$. Consider the shortest path tree rooted at s. Then, there must exist two vertices x and y such that x is the parent of y in the tree, $\delta_x \geq d_x^*$ and $\delta_y < d^*y$. But we know that $\delta_y = \delta_x + w(x,y)$, so this implies that $d_y^* > \delta_y = \delta_x + w(x,y) \geq d_x^* + w(x,y)$, a contradiction.

3. (20 points) Complementary slackness

Problem 29-2 of text.

Answer:

(a) Follows from elementary calculations using the solutions for the primal and dual given in the text.

(b) Let M be the set of j for which $\bar{x}_j > 0$ and N be the set of i for which $\bar{y}_i > 0$. Then, we have

$$
\begin{aligned}
\sum_j c_j \bar{x}_j &= \sum_{j \in M} c_j \bar{x}_j \\
&\leq \sum_{j \in M} \sum_i a_{ij} \bar{y}_i \bar{x}_j \\
&= \sum_{i \in N} \left(\sum_{j \in M} a_{ij} \bar{x}_j \right) \bar{y}_i \\
&\leq \sum_{i \in N} b_i \bar{y}_i \\
&= \sum_i b_i \bar{y}_i
\end{aligned}
$$

By strong duality, the two inequalities should be equalities. The only way for this to hold is to have term-by-term equality $c_j = \sum_i a_{ij} \bar{y}_i$ for $j \in M$ and $b_i = \sum_j a_{ij} \bar{x}_j$ for $i \in N$, yielding the complementary slackness conditions.

(c) Suppose $\bar{x}$ is an optimal solution to the primal. Since the primal is feasible and bounded, so is the dual. So we have an optimal solution $\bar{y}$ for the dual. This, together with part (b), establishes the three desired properties.

Suppose we have a feasible solution satisfying the three properties. Going through the calculations as in part (b), we obtain that $\sum_j c_j \bar{x}_j = \sum_i b_i \bar{y}_i$. This, together with weak duality, implies that $\bar{x}$ and $\bar{y}$ are optimal primal and dual solutions, respectively.

4. (3 + 7 = 10 points) Markov chains and LP duality

A Markov chain is a set of states (of a system) and the probability, for each pair of states, of transitioning from one state to the other. Markov chains have a wide range of applications including as models for several physical or biological processes, in economics and the social sciences, web search, and statistics.

Formally, a Markov chain is an $n \times n$ matrix P where p_{ij} denotes the probability of transitioning from state i to state j; if the system is in state i at time t, then the probability that it is in state j at time $t+1$ is p_{ij}. Clearly, P satisfies the property that the sum of the entries along any row is 1.

One nice, and very useful, property of Markov chains is the existence of a stationary distribution π over the state space, where π is an $n \times 1$ vector with π_i being the probability of the system being in state i. We say that π is stationary if the following holds: if at any time t, π gives the probability distribution for the state of the given system, then π is also the probability distribution for the state in time $t+1$.

For any Markov chain matrix, the existence of a stationary distribution π can be shown easily using LP duality, as we establish in this exercise.

(a) Consider the following LP (over variables π_i) whose constraints define the stationary distribution.

$$
\begin{array}{rcl}
\min & \sum_i \pi_i & \\
\left(\sum_j p_{ji}\pi_j\right) - \pi_i & = & 0 \ \forall i \\
\sum_j \pi_j & \geq & 1 \\
\pi_i & \geq & 0 \ \forall i
\end{array}
$$

Show that the above LP is feasible if and only if the Markov chain has a stationary distribution.

Answer: If the LP is feasible, there exists a nonnegative π such that $P\pi = \pi$. Since $\sum_j \pi_j \geq 1$, we can set $\pi' = \pi/(\sum_j \pi_j)$ to obtain a stationary distribution satisfying $P\pi' = \pi'$. If the Markov chain has a stationary distribution π^* then, $\pi_i = \pi_i^*$ is a feasible solution to the above LP.

(b) Derive the dual for the above LP. Analyze the dual and argue using duality that the primal is always feasible.

Answer: The dual for this LP is the following.

$$
\begin{array}{rcl}
\max & \lambda & \\
\sum_i p_{ij} v_i - v_j + \lambda & \leq & 1 \\
\lambda & \geq & 0
\end{array}
$$

We construct a feasible solution to the dual as follows. Set $v_i = 1$ for all i. For this solution, we get the first inequality of the dual to be $\lambda \leq 1$ since the sum of the entries in each column of P is 1. We set $\lambda = 1$ to obtain a feasible dual solution. It can also be shown that the dual is bounded; its value can be no more than 1 since the dual inequality corresponding to the smallest v_j will yield $\lambda \leq 1$ (using the fact that the sum of the entries in each column is 1). By weak duality, we have a primal optimal solution with $\sum_j \pi_j \leq 1$. But this implies that in the optimal solution $\sum_j \pi_j = 1$. And we obtain that $P\pi = \pi$.