Problem Set 6 (due Wednesday, December 3)

1. ($4 \times 5 = 20$ points) NP-completeness

Problem 34-2 of text.

2. ($5 \times 3 = 15$ points) Maximum-weight spanning tree

Problem 35-6 of text.

3. ($5 \times 3 = 15$ points) NP-completeness and approximation algorithms

Let G be an directed graph with k start nodes s_1 through s_k and k end nodes t_1 through t_k.

(a) Give a reduction from 3-SAT to show that it is NP-hard to determine whether there exist k paths, the ith path from s_i to t_i, $1 \leq i \leq k$, such that no two paths share an edge.

We next consider an optimization version of the above problem. Here, we allow paths to share edges, and define the *load* $\ell(e)$ on an edge e to be the number of s_i-t_i paths that use e. The optimization problem then is to determine a set of s_i-t_i paths that minimizes $\max_e \ell(e)$.

(b) Write an integer linear program for the above problem. (*Hint:* You can view each s_i-t_i path as a flow of unit 1 from s_i to t_i.)

(c) Develop a randomized rounding algorithm which proceeds as follows: Relax the integrality constraint, and solve the LP; Decompose each s_i-t_i flow into a set of paths (we have seen this in a POW in class); Use randomized rounding to select a path. Fill in the details.

The next step is to show that the above rounding algorithm will achieve a load within an $O(\log n)$-factor of the optimal with high probability, say at least $1 - 1/n$, where n is the number of nodes in the graph.

(d) Show that the expected load of an edge e is equal to the total flow on the edge e in the LP solution.

(e) Using Chernoff bounds (described below) show that with probability at least $1 - 1/n$, the load of the path collection is within an $O(\log n)$ factor of the optimal achievable load.

Chernoff bound: Let $X_1, X_2, \ldots, X_n$ be n independent random variables each taking a value of 0 or 1. Let X denote $\sum_i X_i$. Then, for any $\delta > 0$, we have the following.

$$\Pr\left[X > (1+\delta)E[X]\right] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{E[X]}.$$

(*Hint:* For each edge, use the Chernoff bound to place a very small upper bound on the probability that the load of the edge exceeds $O(\log n)$ times the expectation. Then use a union bound to argue that with high probability, the load on every edge is within an $O(\log n)$ factor of the optimal.)