Vectors, Matrices, Rotations

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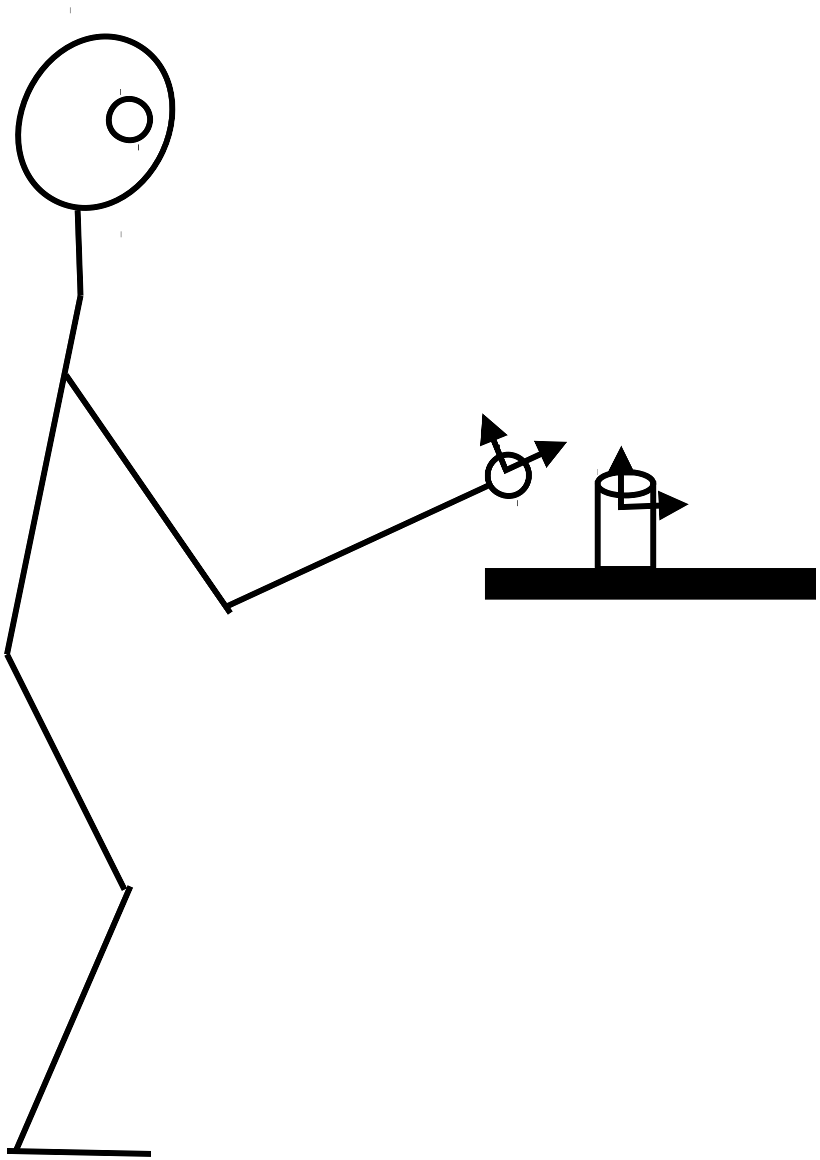
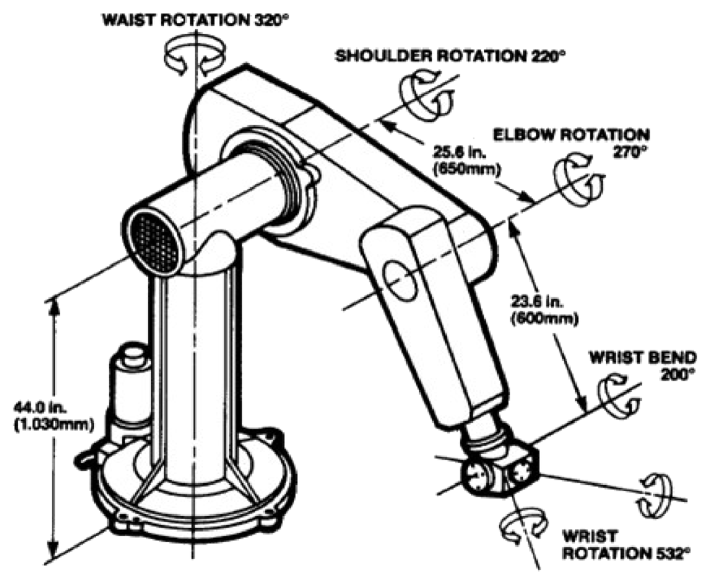
Why are we studying this?



You want to put your hand on the cup…

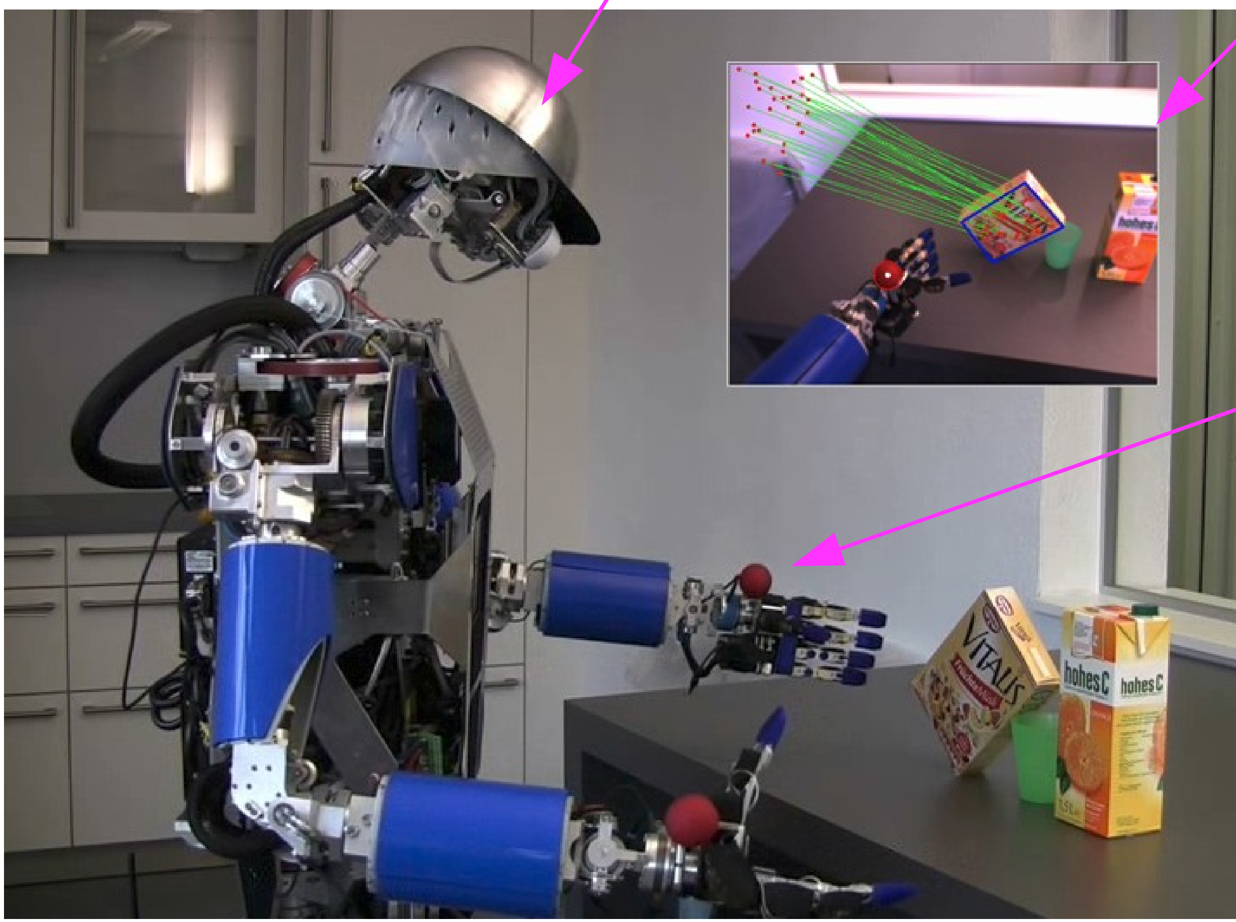
- Suppose your eyes tell you where the mug is and its orientation in the robot *base frame* (big assumption)
- In order to put your hand on the object, you want to align the coordinate frame of your hand w/ that of the object
- This kind of problem makes representation of pose important...



Why are we studying this?

Joint encoders tell us head angle

Visual perception tells us object position and orientation (pose)

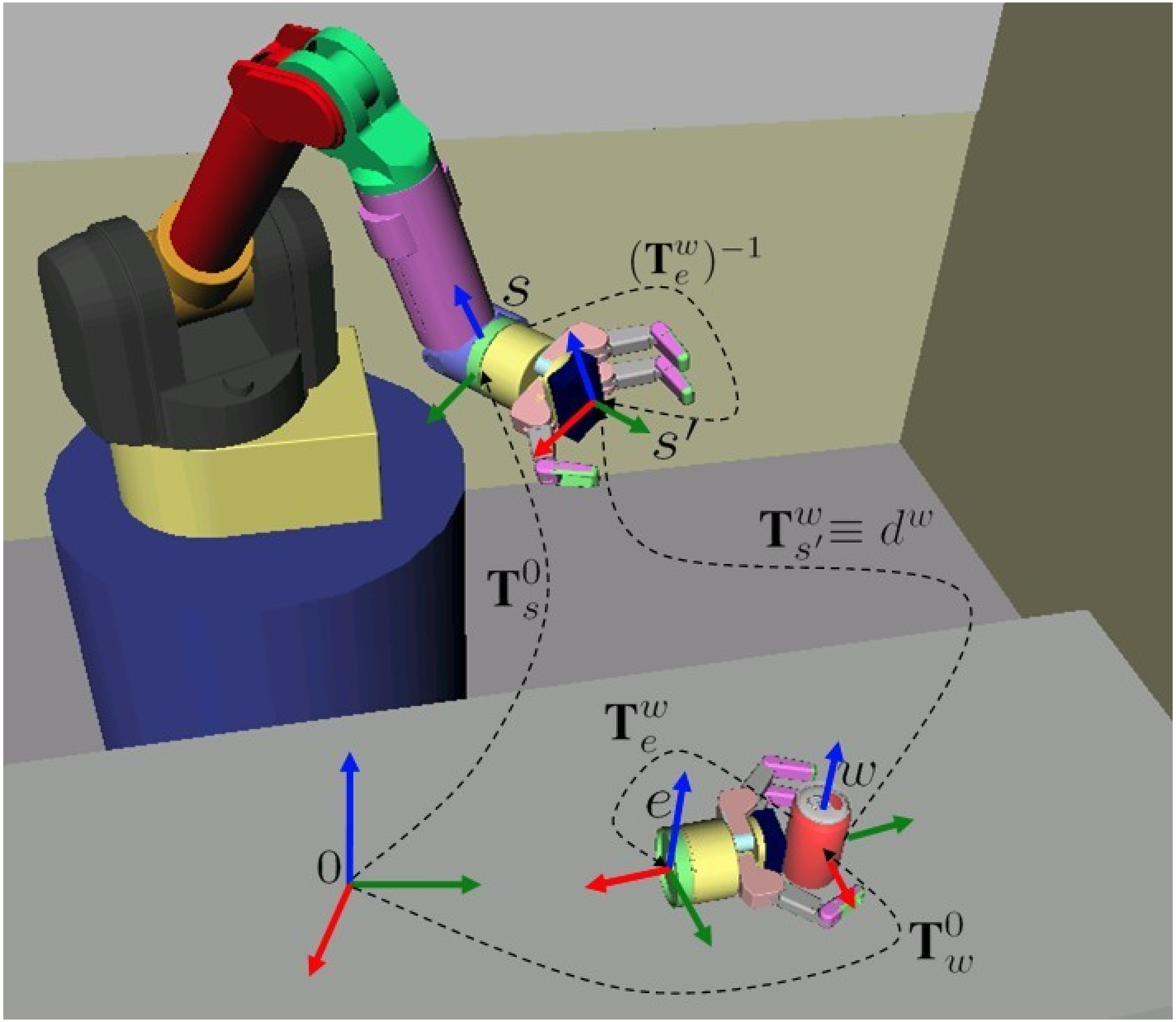


KIT Humanoid

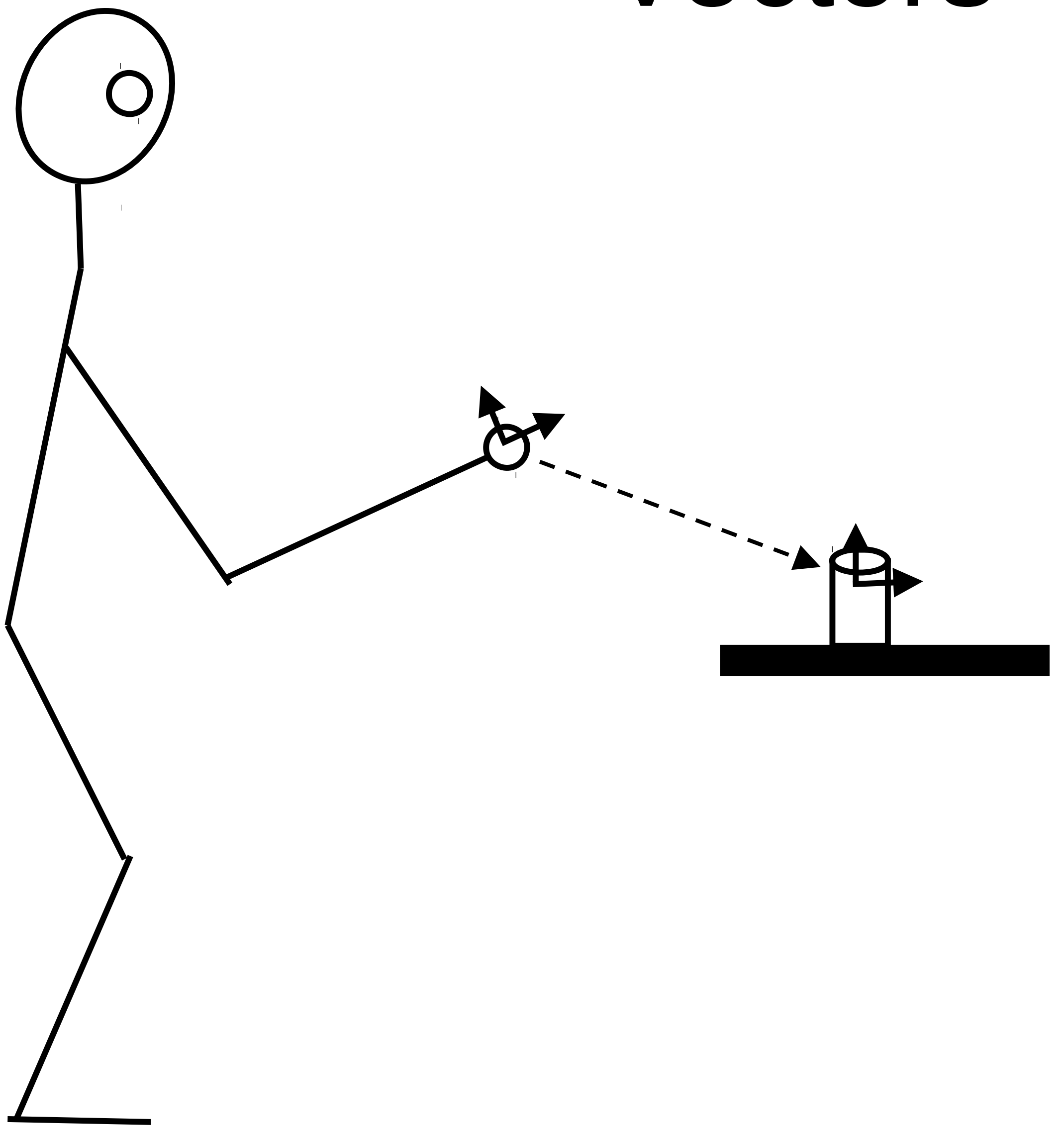
Need to know where hand is...

Need to tell the hand where to move!

Why are we studying this?



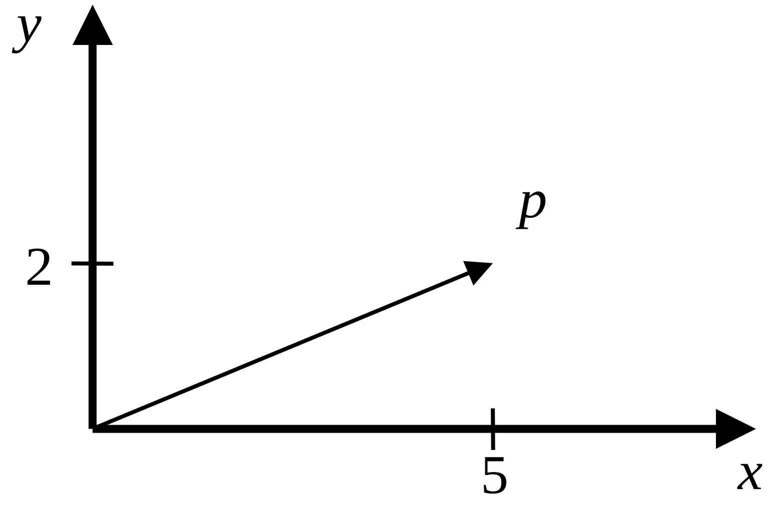
Representing Position: Vectors



Representing Position: vectors

$$p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

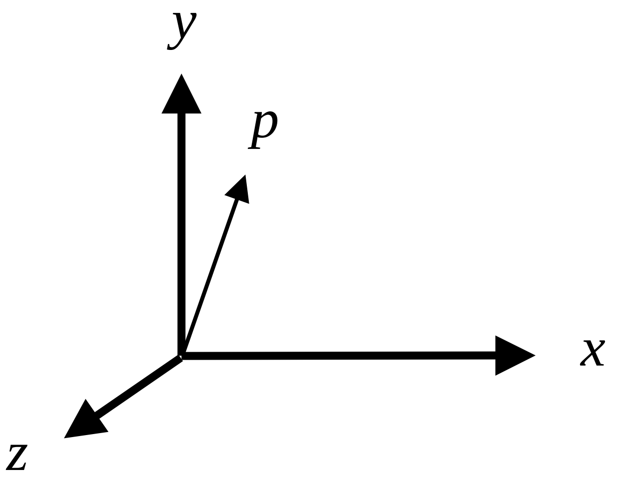
("column" vector)



$$p = (5, 2)$$

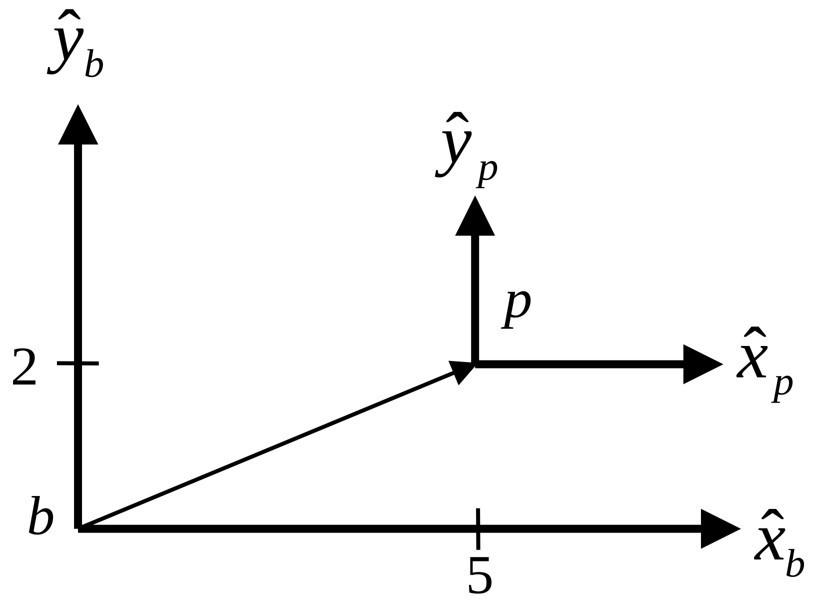
("row" vector)

$$p = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

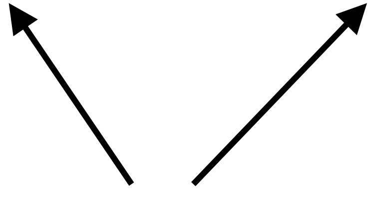


Representing Position: vectors

- Vectors are a way to transform between two different reference frames w/ the same orientation
- The prefix superscript denotes the reference frame in which the vector should be understood



$$^{b}p = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad ^{p}p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

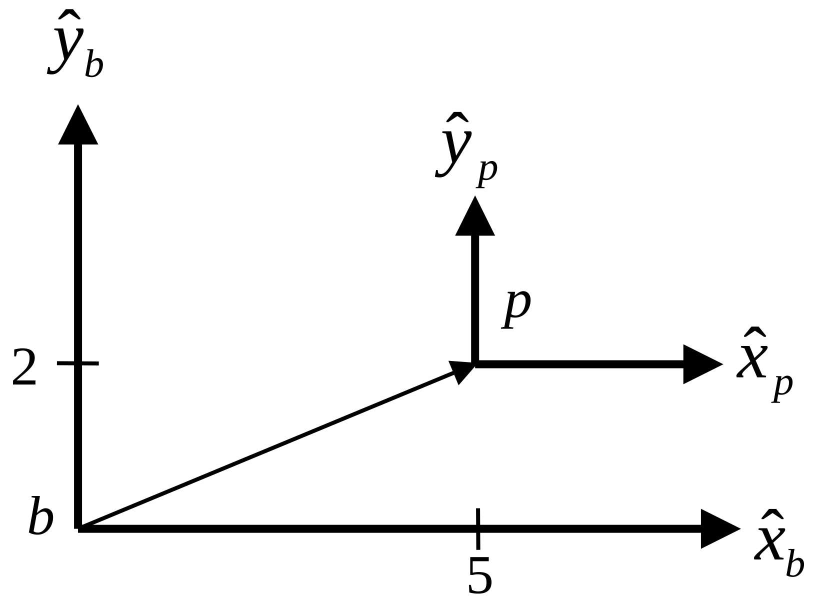


Same point, two different reference frames

Representing Position: vectors

- Note that I am denoting the axes as *orthogonal* unit basis vectors

This means "perpendicular"



$\hat{x}_b$ ← A vector of length one pointing in the direction of the base frame *x* axis

$\hat{y}_b$ ← *y* axis

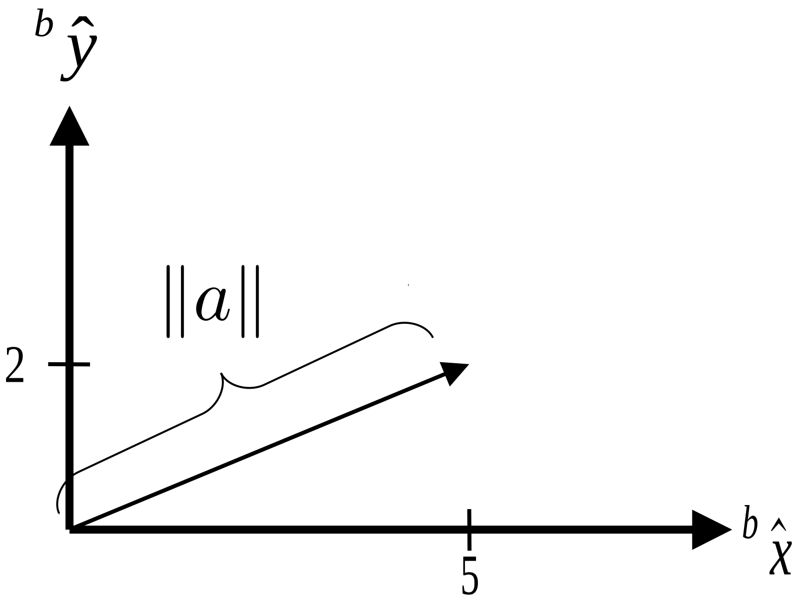
$\hat{y}_p$ ← *p* frame y axis

What is a unit vector?

These are the elements of *a*: $a = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$

Vector length/magnitude:

$$\|a\| = \sqrt{a_x^2 + a_y^2}$$



Definition of unit vector: $\|\hat{a}\| = 1$

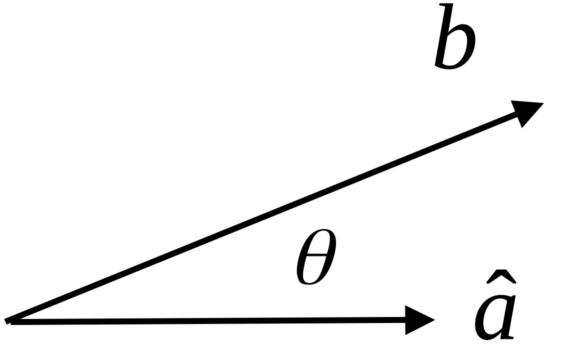
How convert a non-zero vector *a* into a unit vector pointing in the same direction?

Orthogonal vectors

First, define the dot product: $a \cdot b = a_x b_x + a_y b_y$

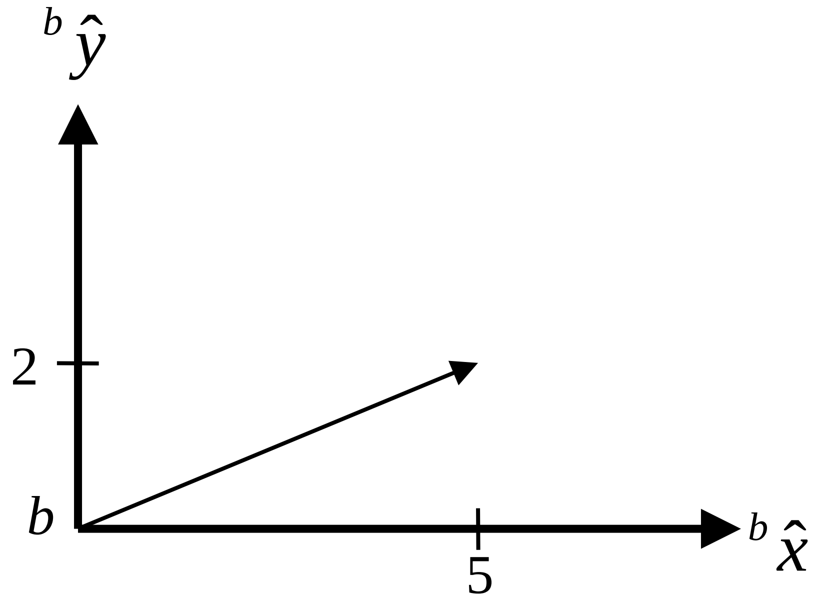
$$= \|a\| \|b\| \cos(\theta)$$

Under what conditions is the dot product zero?

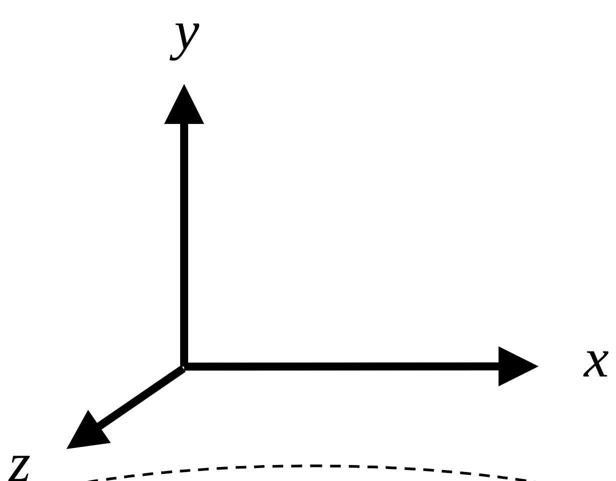


A couple of other random things

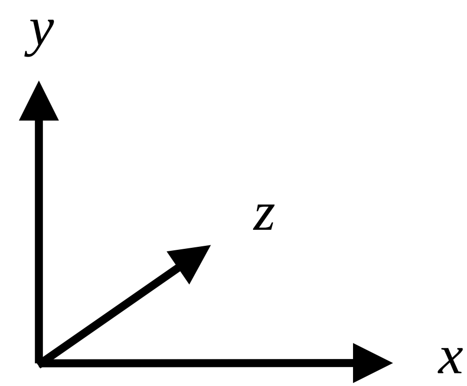
$$p_b = 5\hat{x}_b + 2\hat{y}_b$$



Vectors are elements of R^n



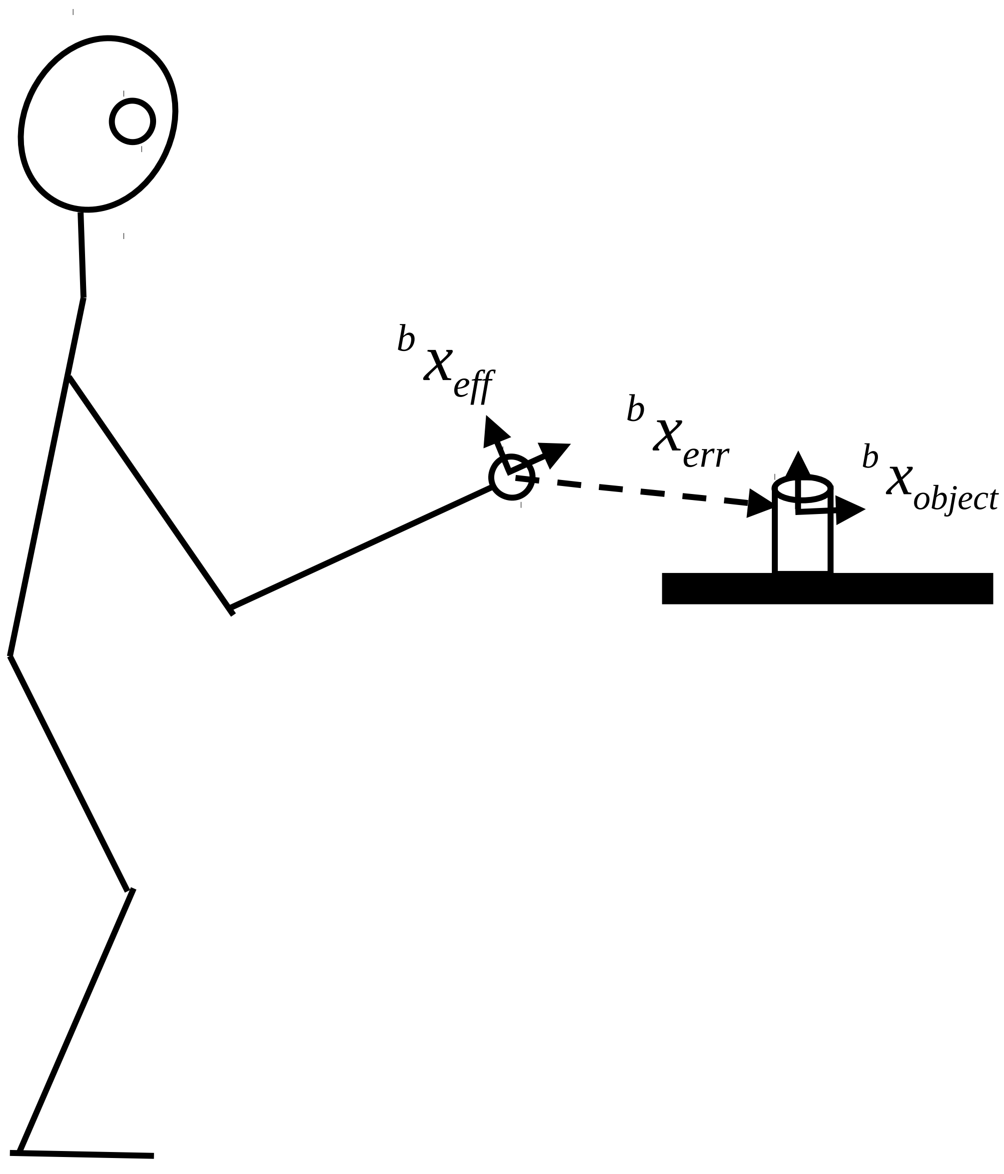
right-handed coordinate frame



left-handed coordinate frame

The importance of differencing two vectors

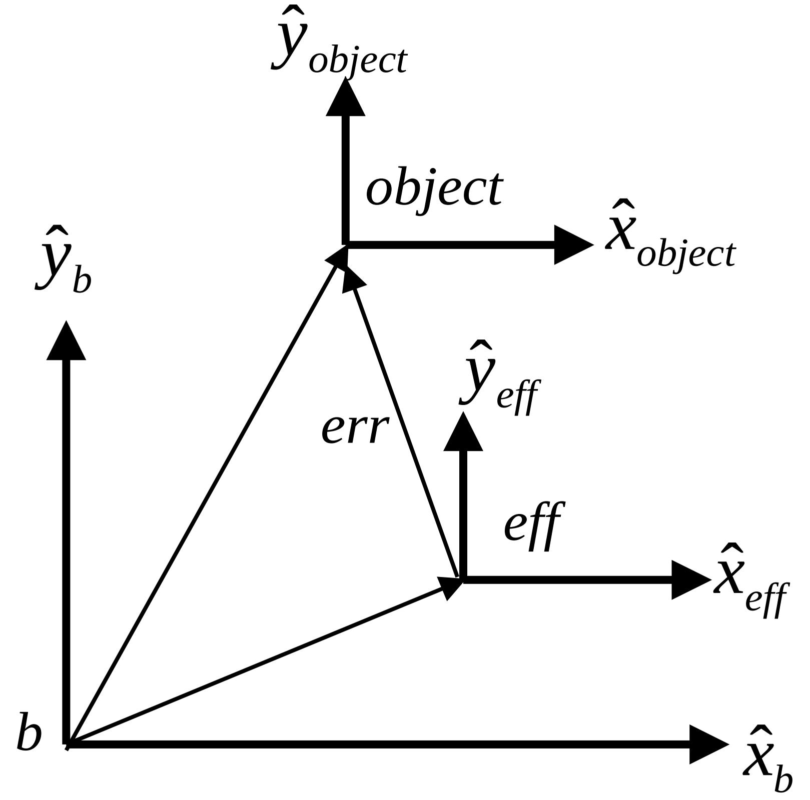
$$ {}^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err} $$



The *eff* needs to make a Cartesian displacement of this much to reach the object

The importance of differencing two vectors

$$ {}^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err} $$



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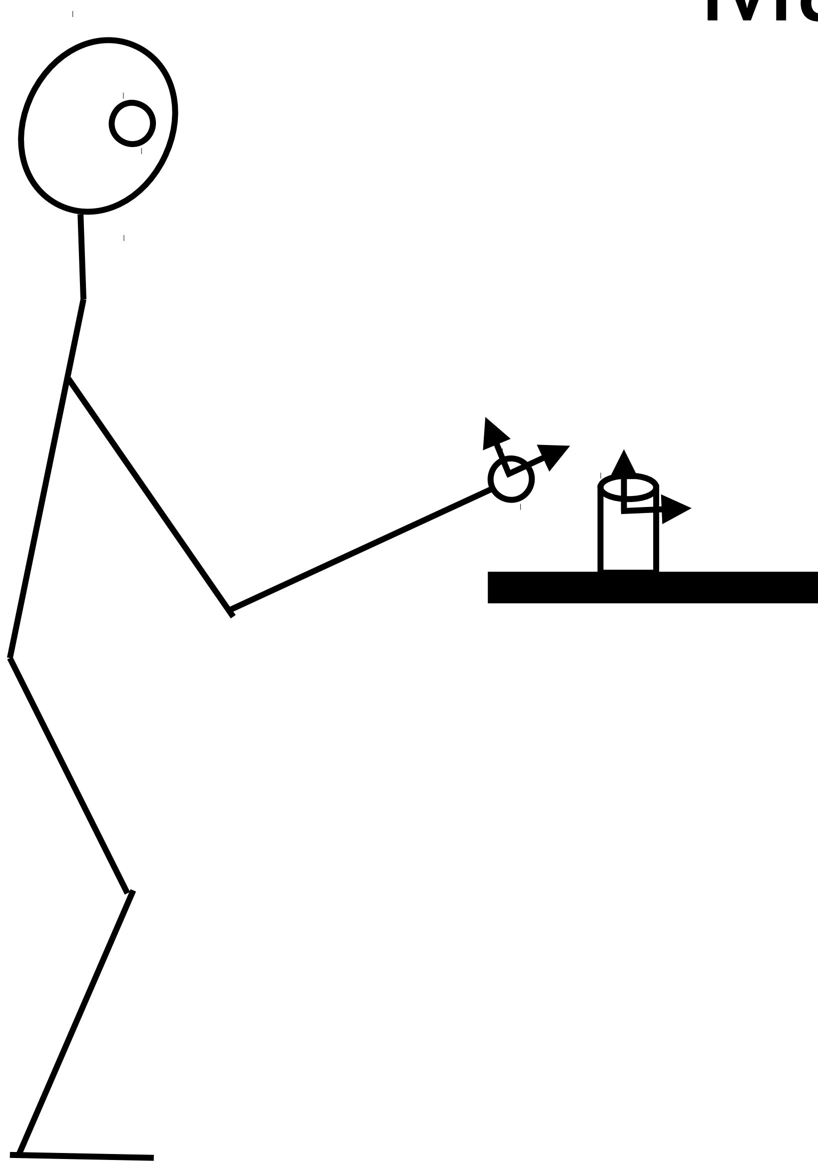
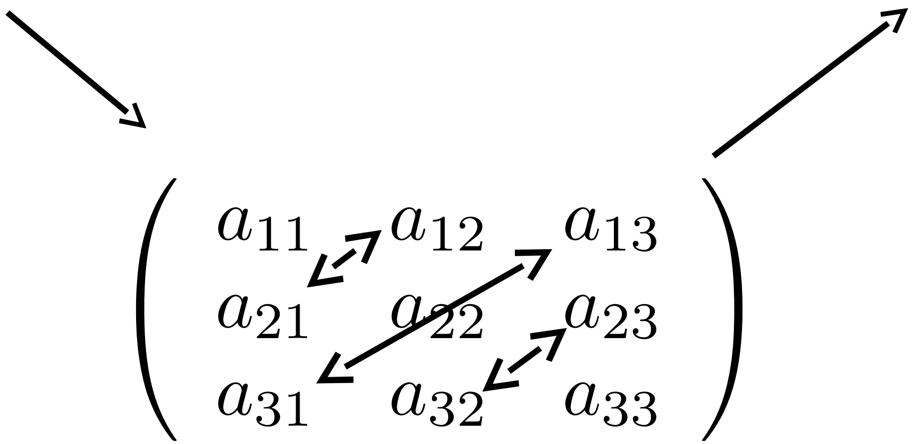
Representing Orientation: Rotation Matrices



- The reference frame of the hand and the object have different orientations
- We want to represent and difference orientations just like we did for positions…

Before we go there – review of matrix transpose

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$



$$p = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \longrightarrow p^T = (5, 2)$$

Question: $(AB)^T = ?$

and matrix multiplication…

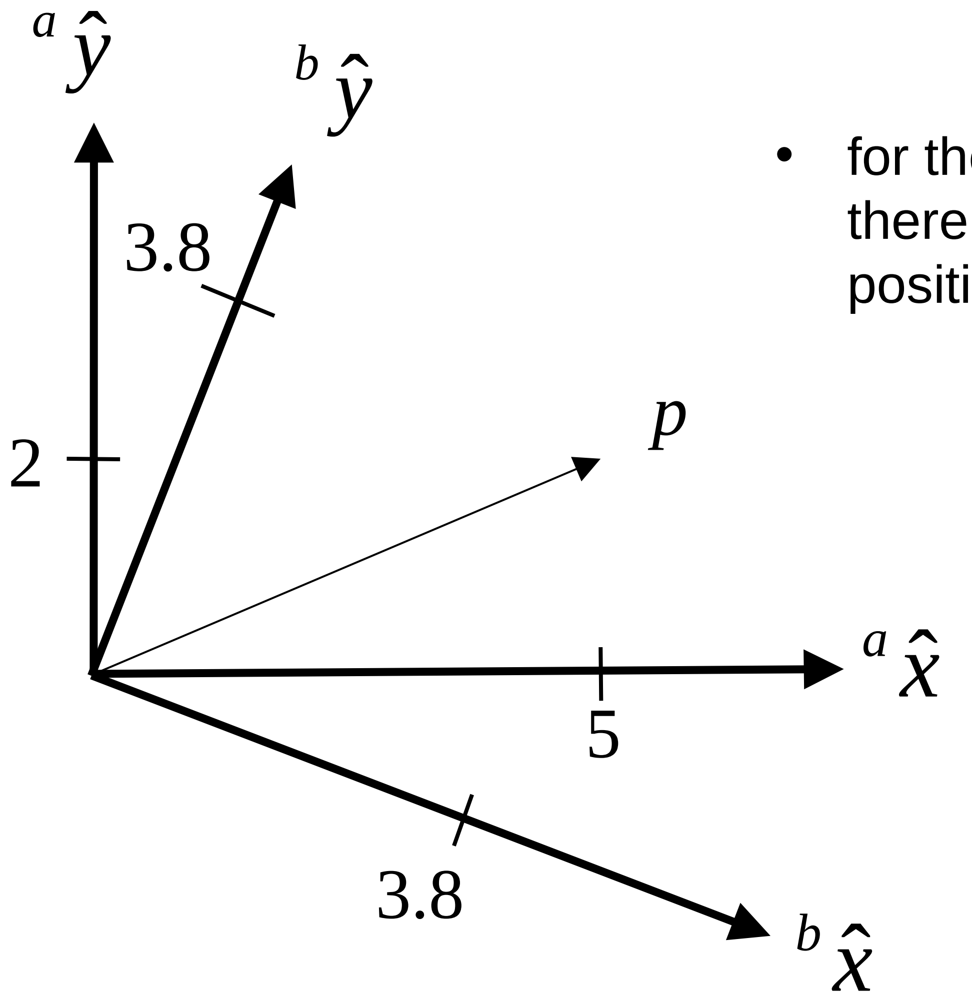
$$A = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \qquad B = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right)$$

$$AB = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) = \left(\begin{array}{cc} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{array} \right)$$

Can represent dot product as a matrix multiply:

$$a \cdot b = a^T b = a_x b_x + a_y b_y$$

Same point - different reference frames



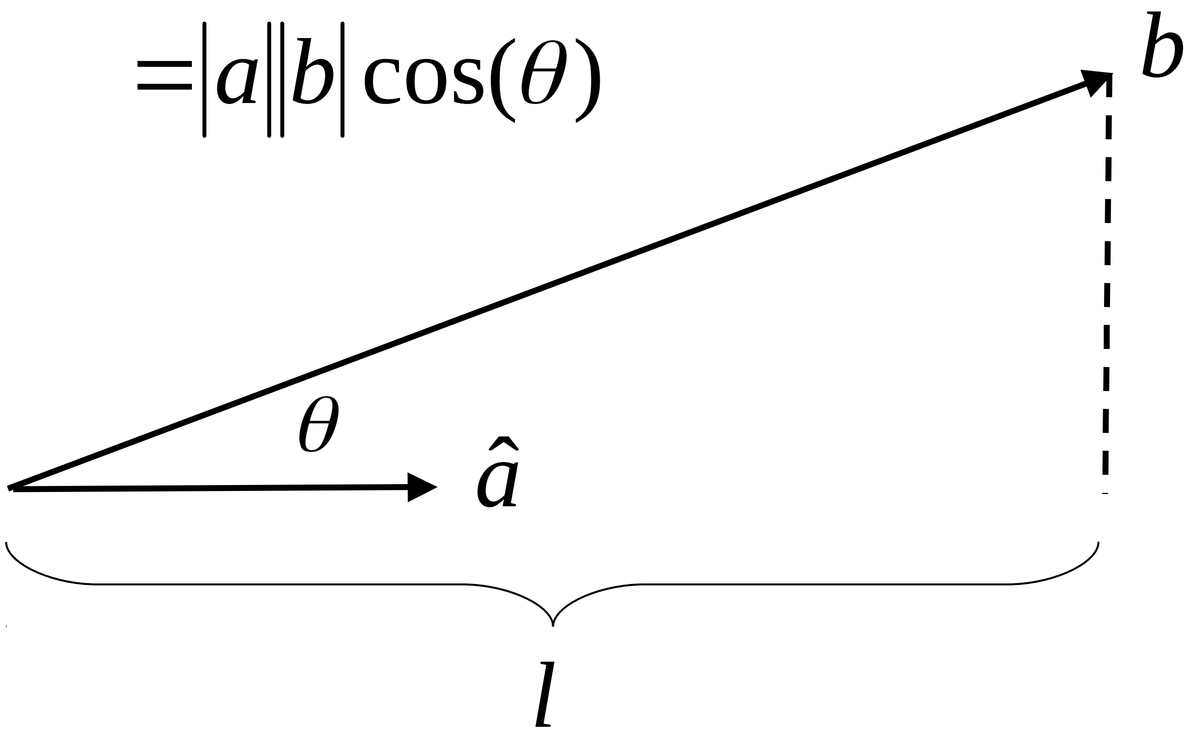
- for the moment, assume that there is no difference in position…

$$^a p = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad ^b p = \begin{pmatrix} 3.8 \\ 3.8 \end{pmatrix}$$

Another important use of the dot product: projection

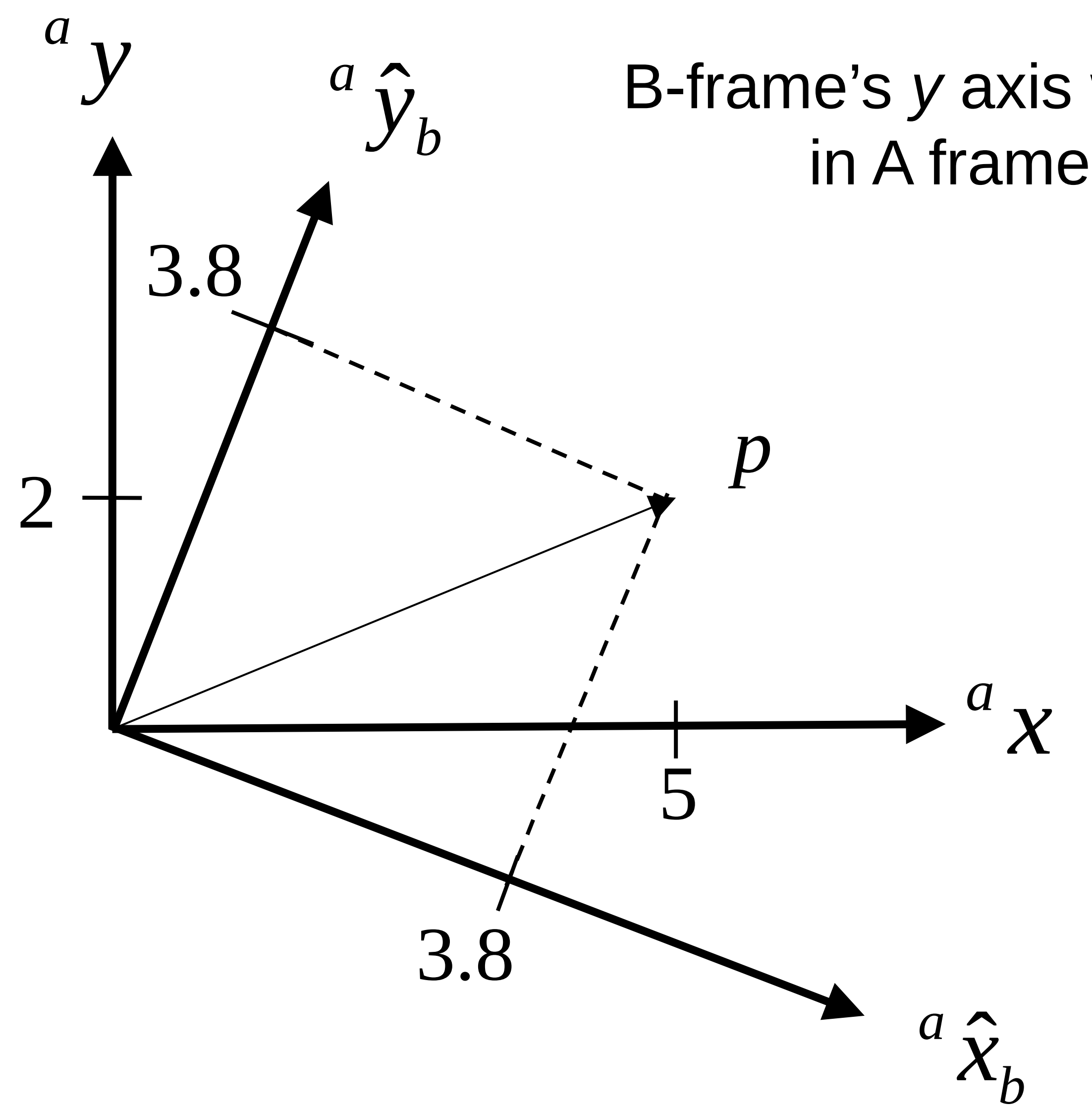
$$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b|\cos(\theta)$$



$$l = \hat{a} \cdot b = |\hat{a}||b|\cos(\theta) = |b|\cos(\theta)$$

Think-pair-share



B-frame's *y* axis written in A frame

Calculate $^b p_x$ given $^a p$

Same point - different reference frames

$$ {}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B^T \; {}^{A}p \\ {}^{A}\hat{y}_B^T \; {}^{A}p \end{pmatrix} = \begin{pmatrix} {}^{A}\hat{x}_B^T \\ {}^{A}\hat{y}_B^T \end{pmatrix} {}^{A}p $$

$$ {}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B^T \\ {}^{A}\hat{y}_B^T \end{pmatrix} {}^{A}p $$

$$ {}^{B}p = {}^{A}R_B^T \; {}^{A}p $$

Where: $$ {}^{A}R_B^T = \begin{pmatrix} {}^{A}\hat{x}_B^T \\ {}^{A}\hat{y}_B^T \end{pmatrix} $$

Rotation matrix

Rotation matrices

A rotation matrix is a 2x2 or 3x3 matrix *R* such that:

1. $R^T R = I$ → Rows and columns are unit length and orthogonal
2. $det(R) = +1$ → Right handed coordinate frame

Rotation matrix inverse equals transpose:

$$R^T R = I$$
$$R = (R^T)^{-1}$$
$$R = (R^{-1})^T$$
$$R^T = R^{-1}$$

Rotation matrices

A rotation matrix is a 2x2 or 3x3 matrix *R* such that:

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$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

Unit vectors and orthogonal to each other

Rotation matrices

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1. $R^T R = I$ → Rows and columns are unit length and orthogonal
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$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

Unit vectors and orthogonal to each other

Rotation matrices

By convention: $^{B}p = {}^{B}R_{A}\ {}^{A}p$

where $^{B}R_{A} = \left({}^{B}\hat{x}_{A}\ {}^{B}\hat{y}_{A} \right)$

Similarly: $^{A}p = {}^{A}R_{B}\ {}^{B}p$

Notice:

$$^{A}R_{B} = ({}^{B}R_{A})^{-1}$$
$$= {}^{B}R_{A}^{T}$$

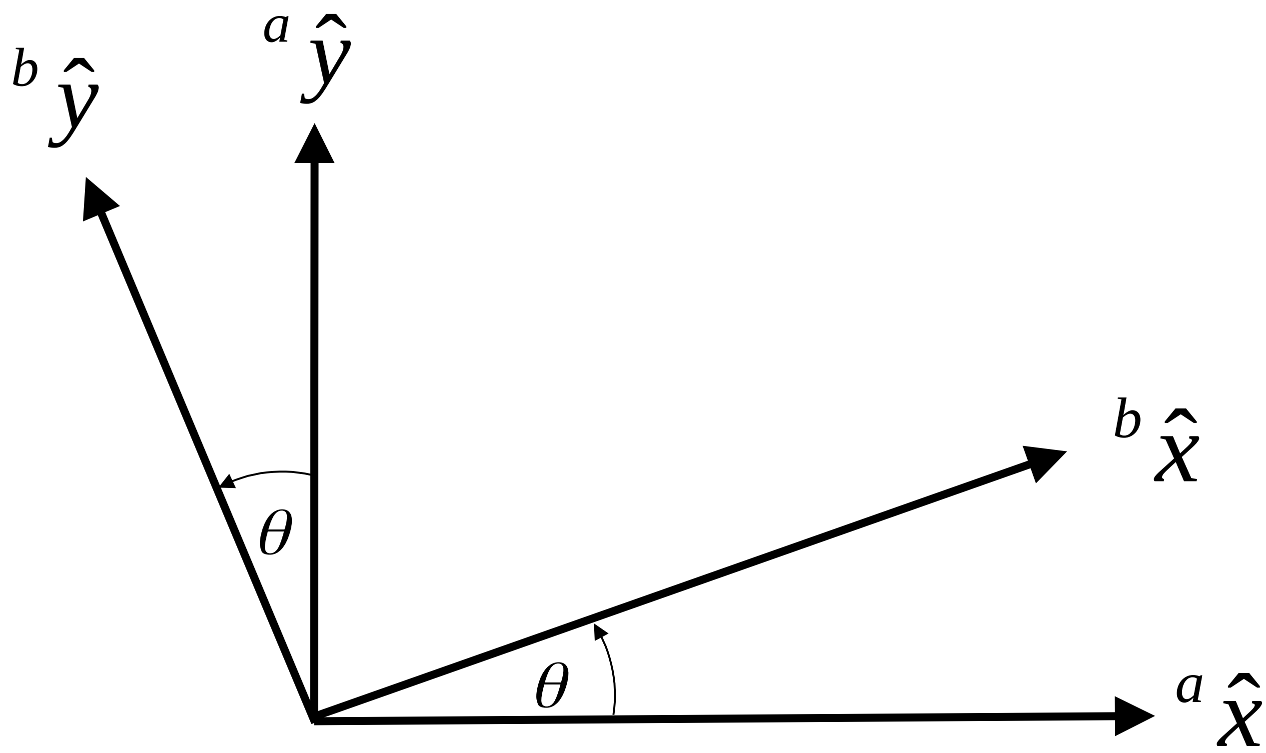
Because of properties of rotation matrix

Think-pair-share

Given: ${}^{B}\hat{x}_{A}, {}^{B}\hat{y}_{A}$

Calculate: ${}^{A}\hat{x}_{B}, {}^{A}\hat{y}_{B}$

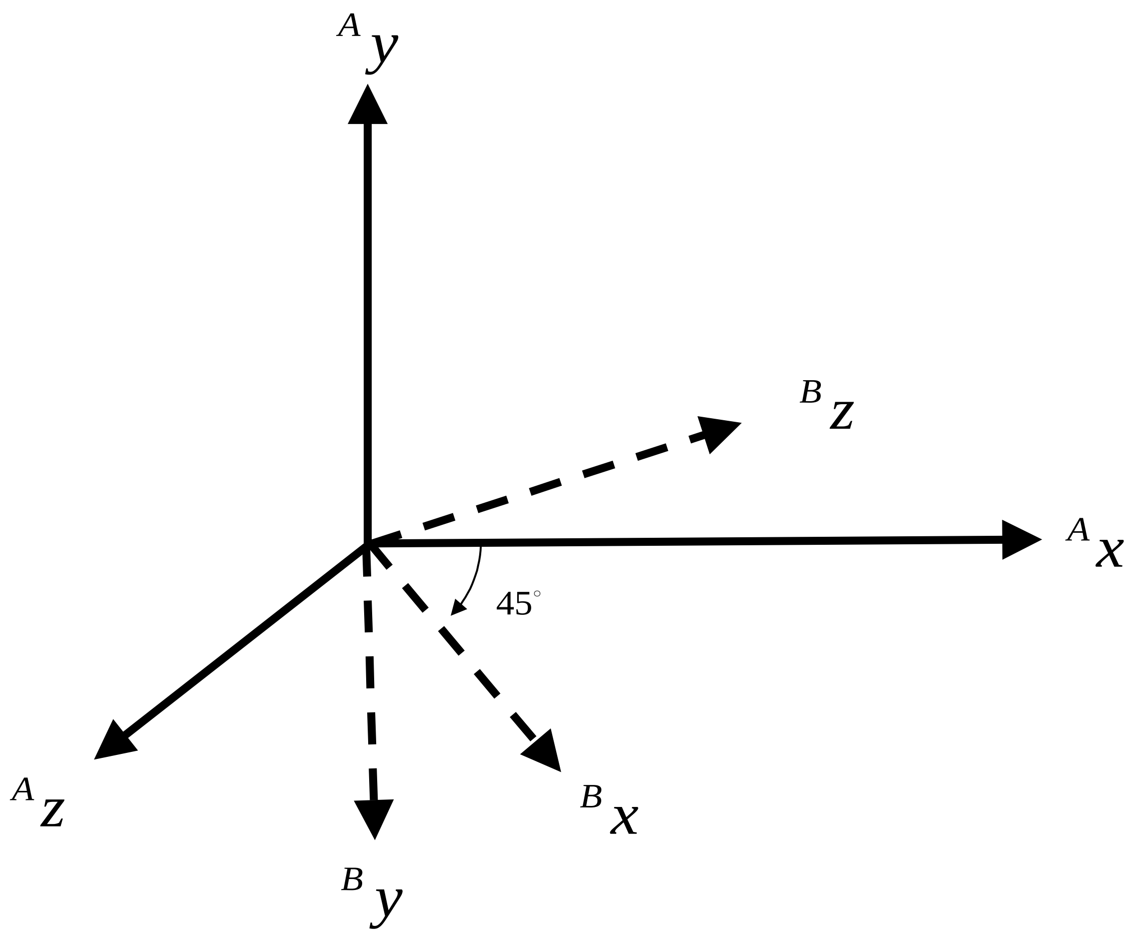
Example 1: rotation matrix



$$ {}^a\hat{x}_b = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad {}^aR_b = \begin{pmatrix} {}^a\hat{x}_b & {}^a\hat{y}_b \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} $$

$$ {}^a\hat{y}_b = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad {}^bR_a = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} $$

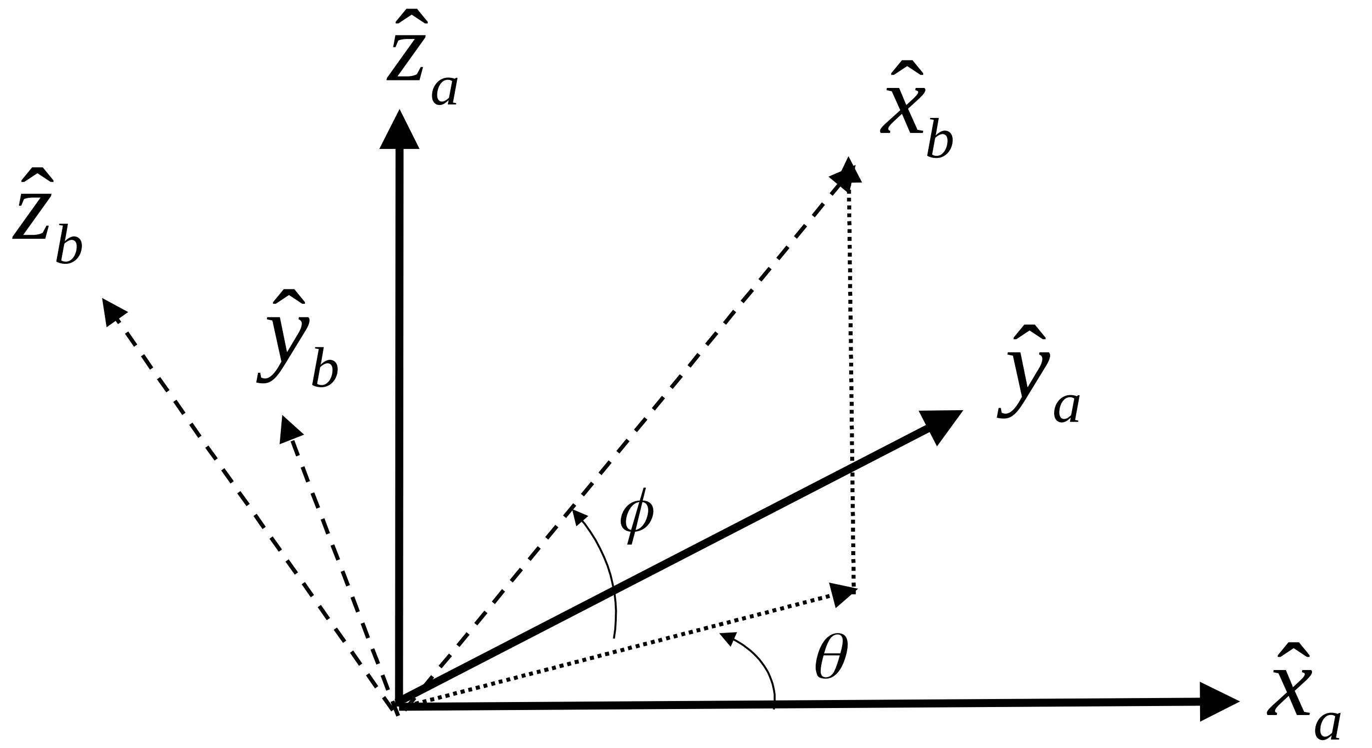
Example 2: rotation matrix



1. Calculate: $^{A}R_{B}$

2. What’s the magnitude of this rotation?

Example 3: rotation matrix



$$
{}^{a}R_{c} = \begin{pmatrix} c_\theta c_\varphi & -s_\theta & c_\theta c_{\varphi+\frac{\pi}{2}} \\ s_\theta c_\varphi & c_\theta & s_\theta c_{\varphi+\frac{\pi}{2}} \\ s_\varphi & 0 & s_{\varphi+\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} c_\theta c_\varphi & -s_\theta & -c_\theta s_\varphi \\ s_\theta c_\varphi & c_\theta & -s_\theta s_\varphi \\ s_\varphi & 0 & c_\varphi \end{pmatrix}
$$

Rotations about x, y, z

$$R_z(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix}$$

These rotation matrices encode the basis vectors of the after-rotation reference frame in terms of the before-rotation reference frame

Remember those double-angle formulas…

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$