Writing Dynamics in State Space Form

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Motivation

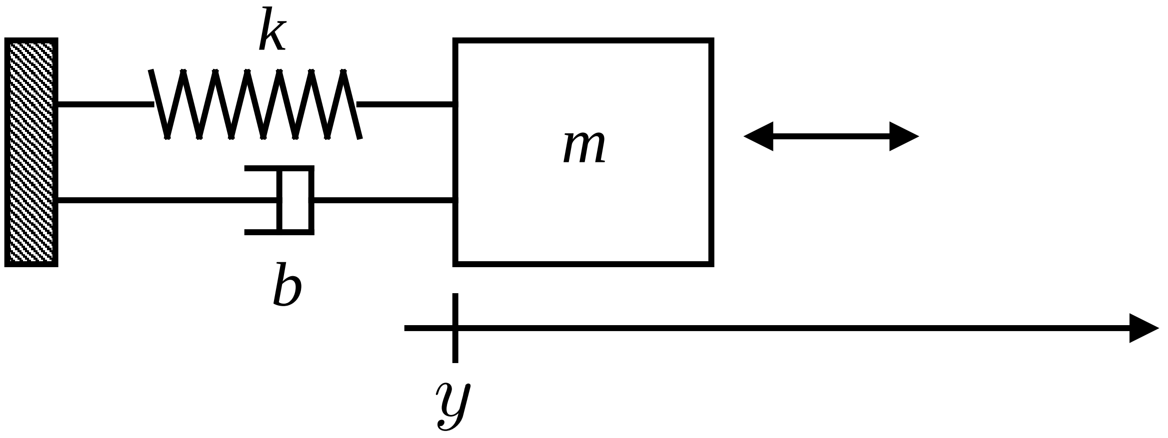
In order to reason about complex dynamical systems, we need to write system dynamics in a convenient form.

How encode dynamics of an inverted pendulum?

How plan walking trajectories?

How plan flying trajectories?

A simple system

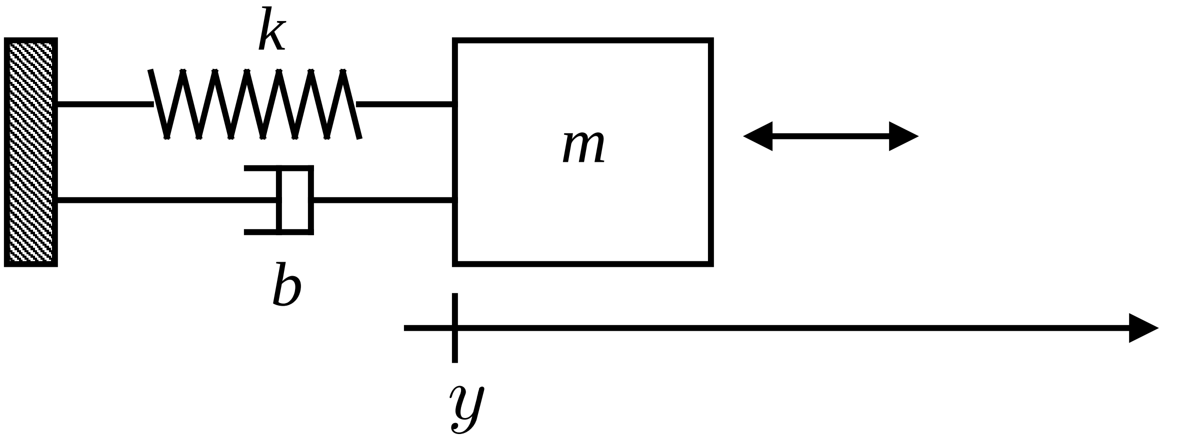


Force exerted by the spring: $f = ky$

Force exerted by the damper: $f = b\dot{y}$

Force exerted by the inertia of the mass: $f = m\ddot{y}$

A simple system



Consider the motion of the mass

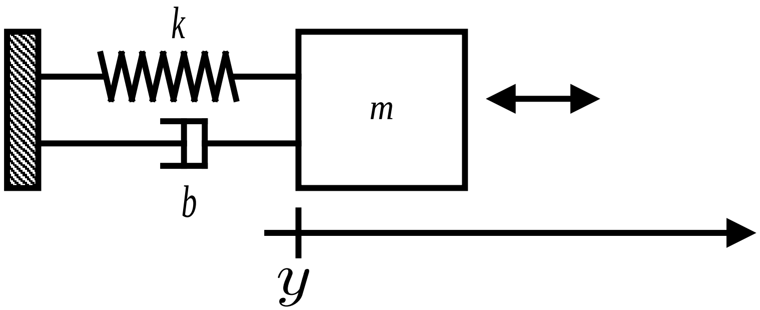
- there are no other forces acting on the mass
- therefore, the equation of motion is the sum of the forces:

$$0 = m\ddot{y} + b\dot{y} + ky$$

This is called a linear system. Why?

A simple system

Let's express this in "state space form":



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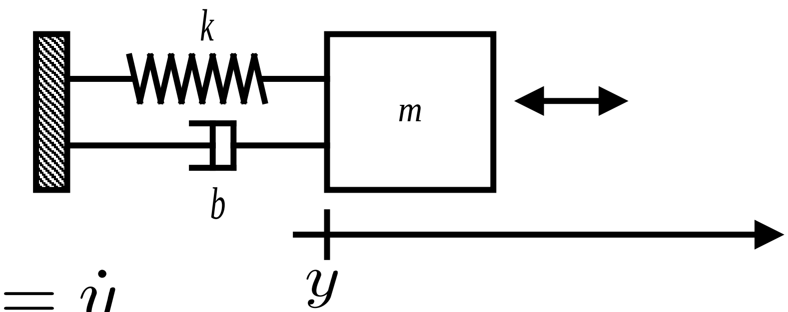

$$0 = m\ddot{y} + b\dot{y} + ky$$

$$\begin{cases} \dot{y} = \dot{y} \\ \ddot{y} = -\dfrac{1}{m}(b\dot{y} + ky) \end{cases}$$

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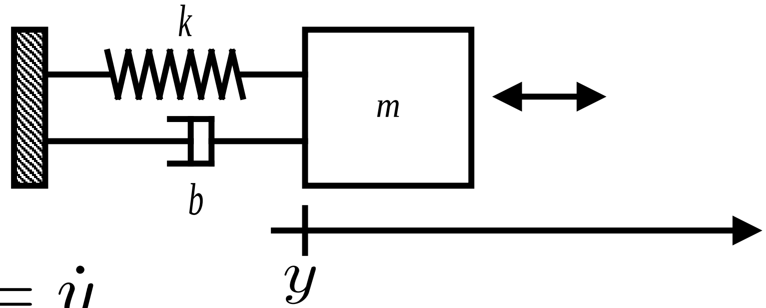

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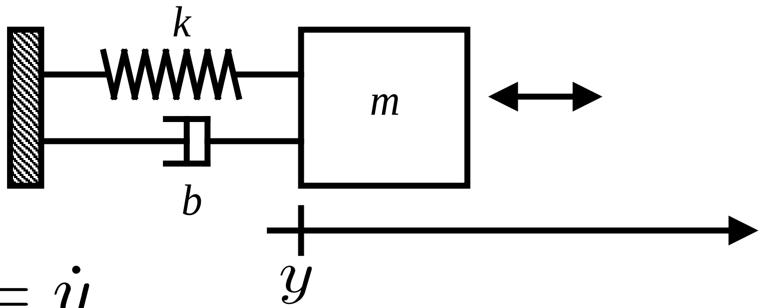

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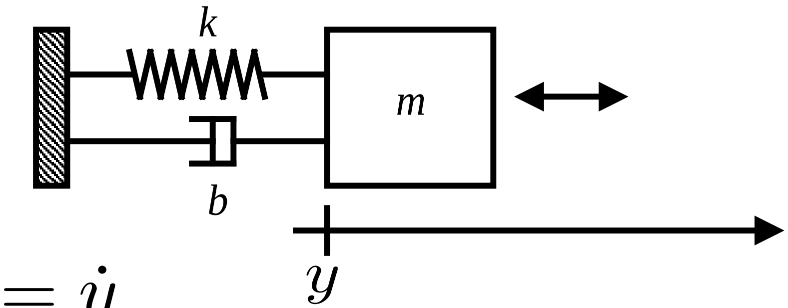

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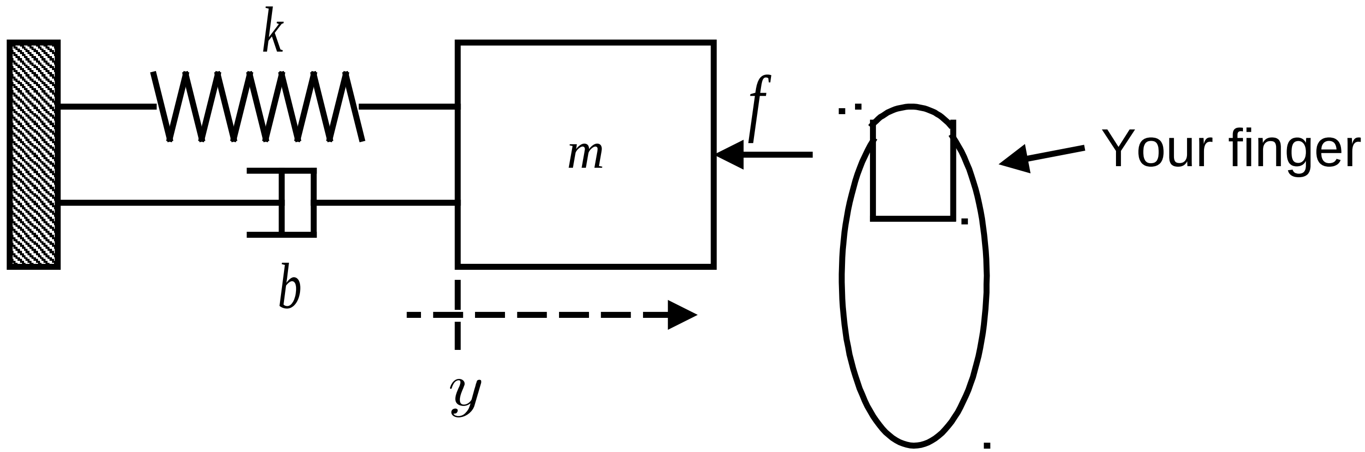
$\dot{x} = Ax$ where $x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

A simple system



Suppose that you apply a force:

$$u = m\ddot{y} + b\dot{y} + ky \quad \Rightarrow \quad \begin{cases} \dot{y} = \dot{y} \\ \ddot{y} = u - \dfrac{1}{m}(b\dot{y} + ky) \end{cases}$$

$$\begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k/m & -b/m \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

A simple system

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$$\Downarrow$$

$$\dot{x} = Ax + Bu$$

A simple system

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Canonical form for a linear system

$$\dot{x} = Ax + Bu$$

Continuous time vs discrete time

$$\dot{x} = Ax + Bu$$ ← Continuous time

$$x_{t+1} = Ax_t + Bu_t$$ ← Discrete time

Continuous time vs discrete time

$$\dot{x} = Ax + Bu$$

Continuous time

$$x_{t+1} = Ax_t + Bu_t$$

Discrete time

What are *A* and *B* now?

Continuous time vs discrete time

$$\dot{x} = Ax + Bu$$

Continuous time

$$x_{t+1} = Ax_t + Bu_t$$

Discrete time

$$\begin{pmatrix} y_{t+1} \\ \dot{y}_{t+1} \end{pmatrix} = A \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} + Bu$$

What are *A* and *B* now?

Simple system in discrete time

We want something in this form: $$\begin{pmatrix} y_{t+1} \\ \dot{y}_{t+1} \end{pmatrix} = A \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} + Bu$$

$$y_{t+1} = y_t + \dot{y}_t dt$$

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Continuous time vs discrete time

$$\dot{x} = Ax + Bu$$ ← CT

$$x_{t+1} = Ax_t + Bu_t$$ ← DT

$$\begin{pmatrix} \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$ CT

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Continuous time vs discrete time

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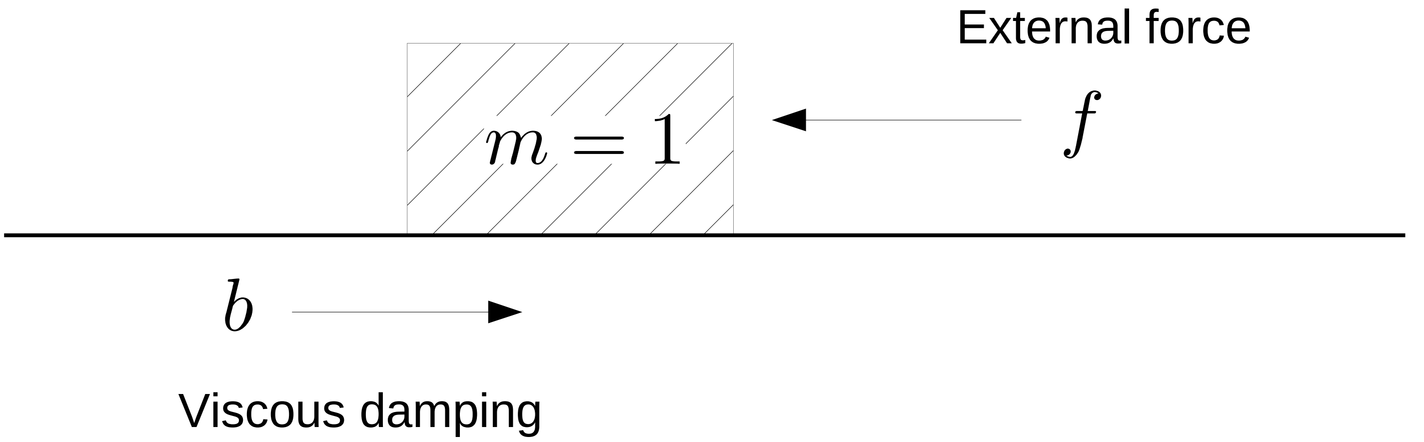
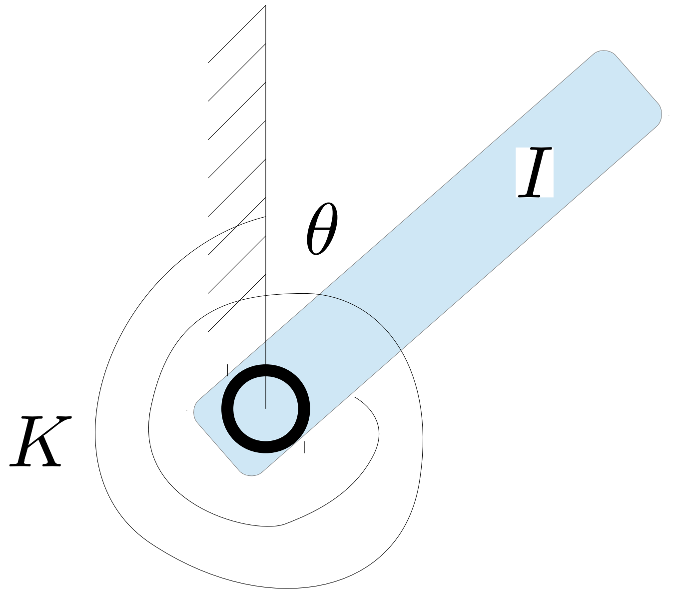
<u>In this class, we're going to focus on discrete time representations...</u>

Think-pair-share



Express DT dynamics of this system in state space form

Think-pair-share



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