Linear MPC

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Linear Model Predictive Control

Drawbacks to LQR: hard to encode constraints
– suppose you have a hard goal constraint?
– suppose you have piecewise linear state and action constraints?

Answer:
– solve control as a new optimization problem on every time step

Linear Model Predictive Control

<u>Given:</u>

System: $$x_{t+1} = Ax_t + Bu_t$$

Cost function: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

where: $$X = (x_1, \ldots, x_T)$$ $$U = (u_1, \ldots, u_{T-1})$$

Initial state: x_1

<u>Calculate:</u> *U* that minimizes J(*X*,*U*)

Linear Model Predictive Control

Given:

System: x ... A ... $+ B$ ...

Cost function: ... $u_t^T R u_t$

We're going to solve this problem
by expressing it explicitly as
a quadratic program

Initial state:

Calculate: *U* that minimizes J(*X*,*U*)

Quadratic program

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

Quadratic program

Constants are part of problem statement:

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

x is the variable

Problem: find the value of x that minimizes the objective subject to the constraints

Quadratic program

Quadratic objective function

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Linear inequality constraints

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

Linear equality constraints

Quadratic program

$$P = P^T \geq 0$$

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

Quadratic program

$$P = P^T \geq 0$$

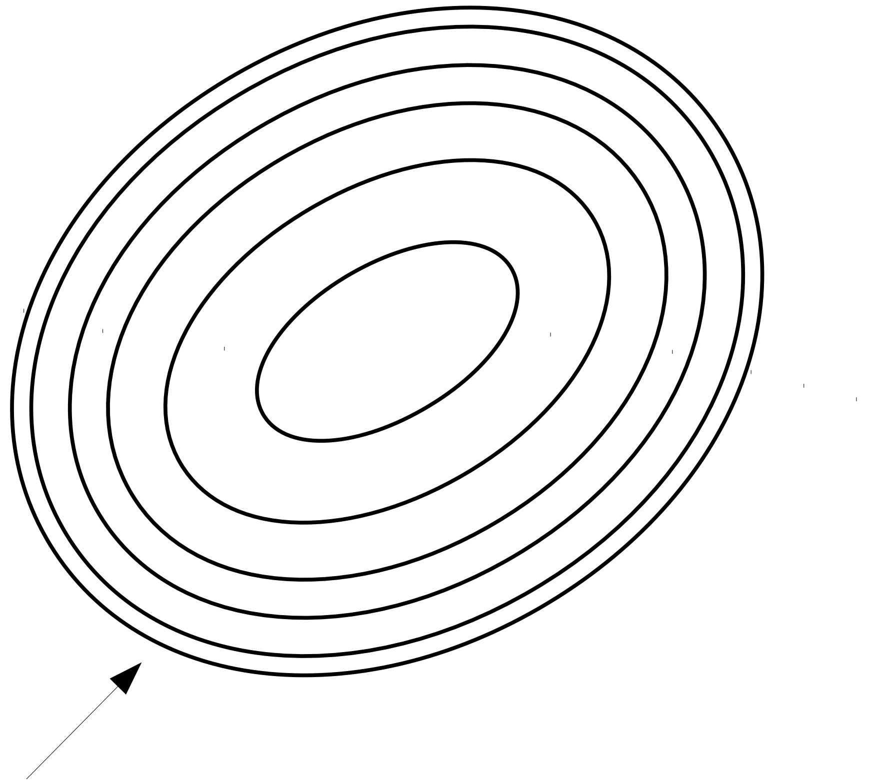
Why?

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

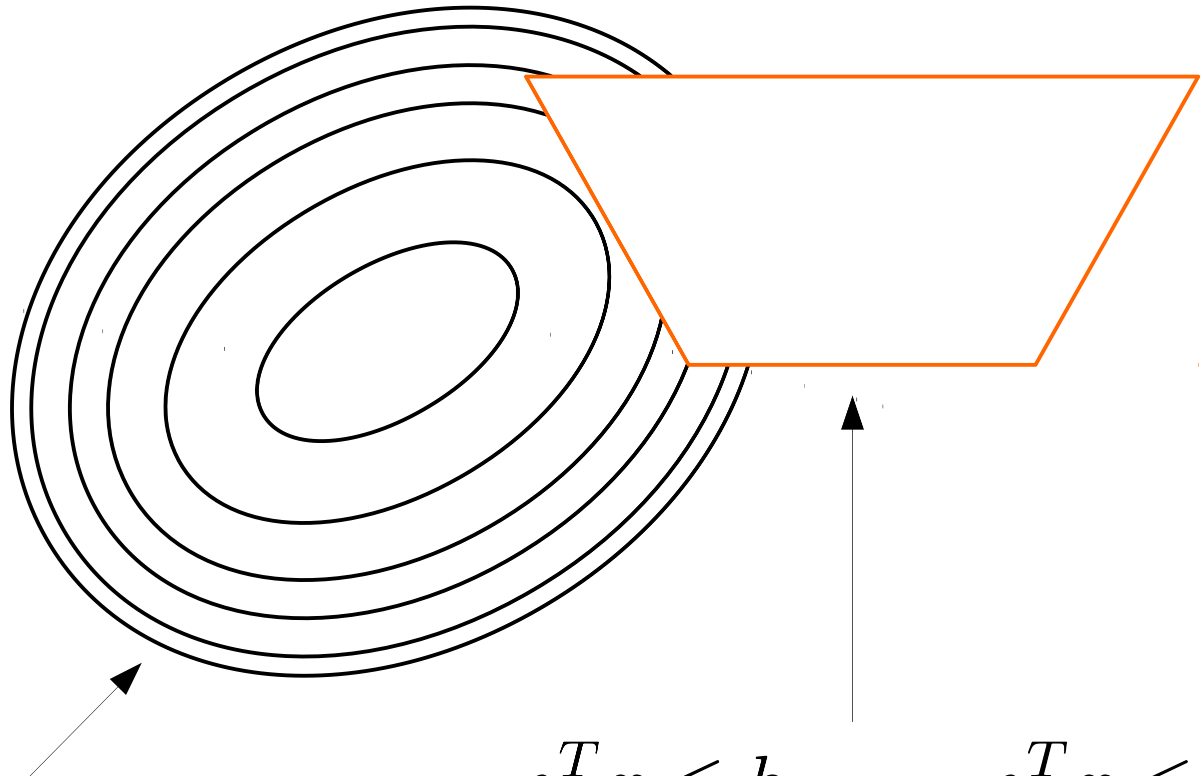
Quadratic program



$$\frac{1}{2}x^T P x + q^T x + r$$

Quadratic objective function

Quadratic program



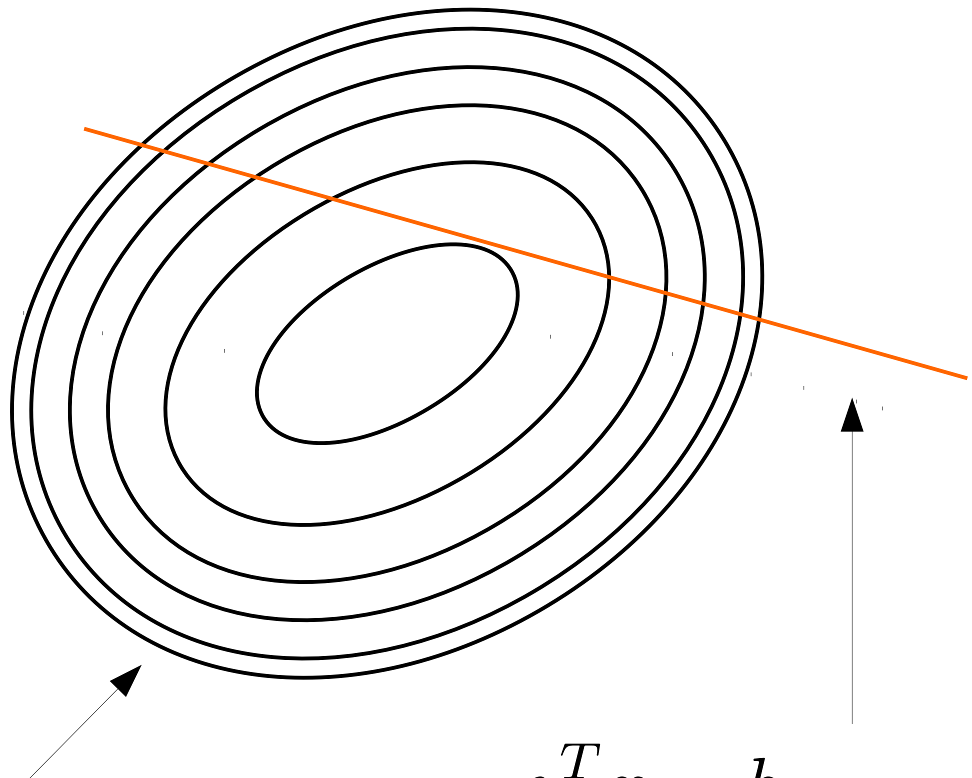
$$\frac{1}{2}x^T P x + q^T x + r$$

Quadratic objective function

$$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

Inequality constraints

Quadratic program



$$\frac{1}{2}x^T P x + q^T x + r$$

Quadratic objective function

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

equality constraints

QP versus Unconstrained Optimization

Original QP

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

QP versus Unconstrained Optimization

Unconstrained version of original QP

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

Subject to: $$g_1^T x \leq h_1, \ldots, g_n^T x \leq h_n$$

$$a_1^T x = b_1, \ldots, a_n^T x = b_n$$

QP versus Unconstrained Optimization

<u>Unconstrained version of original QP</u>

Minimize: $$\frac{1}{2}x^T P x + q^T x + r$$

How do we minimize this expression?

QP versus Unconstrained Optimization

Unconstrained version of original QP

Minimize:

$$\frac{1}{2}x^T P x + q^T x + r$$

How do we minimize this expression?

$$\frac{\partial \left[\frac{1}{2}x^T P x + q^T x + r\right]}{\partial x} = 0$$

$$x = -P^{-1}q$$

Linear Model Predictive Control

Minimize: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Subject to: $$x_{t+1} = Ax_t + Bu_t$$

$$x_1 = \text{start state}$$

$$x_T = \text{goal state}$$

Linear Model Predictive Control

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What are the variables?

Linear Model Predictive Control

Minimize: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Subject to: $$x_{t+1} = Ax_t + Bu_t$$

$$x_1 = \text{start state}$$

$$x_T = \text{goal state}$$

What other constraints might we want add?

Linear Model Predictive Control

Minimize: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Subject to: $$x_{t+1} = Ax_t + Bu_t$$

$$x_1 = \text{start state}$$

$$x_T = \text{goal state}$$

$$|\dot{y}_t| \leq c$$

$$\dot{y}_{20} = 0$$

Linear Model Predictive Control

Minimize: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Subject to: $$x_{t+1} = Ax_t + Bu_t$$

$$x_1 = \text{start state}$$

$$x_T = \text{goal state}$$

$$|\dot{y}_t| \leq c$$

$$\dot{y}_{20} = 0$$

Can't express these constraints in standard LQR

Linear MPC Receding Horizon Control

Re-solve the quadratic program on each time step:
– always plan another T time steps into the future

Minimize: $$J(X,U) = x_T^T Q_T x_T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Subject to: $$x_{t+1} = A x_t + B u_t$$

$$x_1 = \text{start state}$$