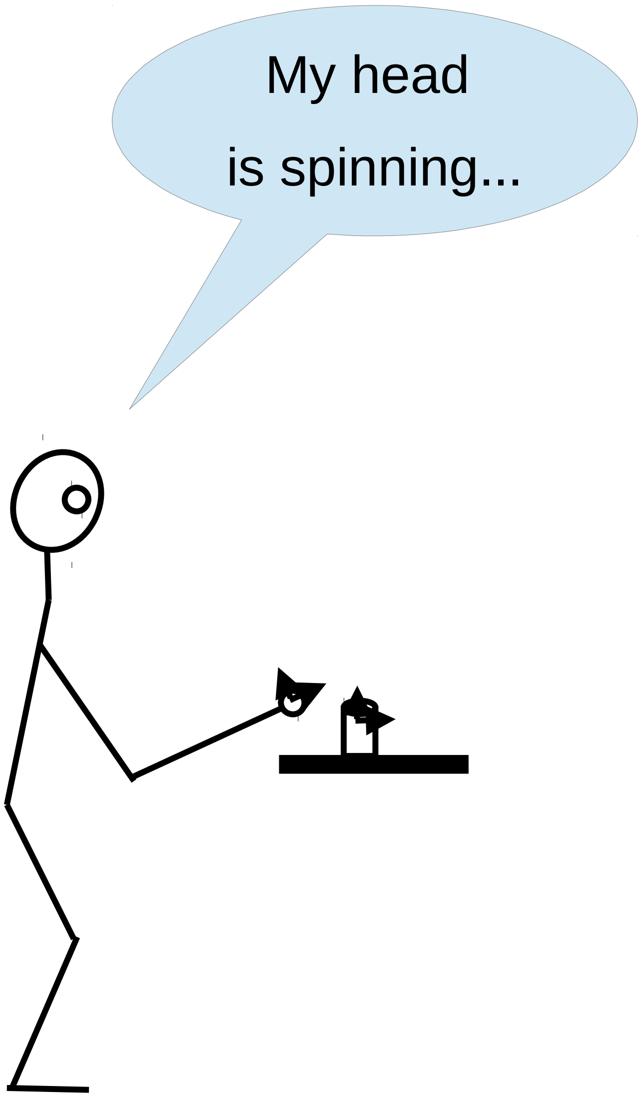
Representing Orientation

Robert Platt
Northeastern University



The space of rotations

$$SO(3) = \{ R \in R^{3\times3} \mid RR^T = I, \det(R) = +1 \}$$

↑

Special orthogonal group(3):

Why $\det(R) = \pm 1$?

Rotations preserve distance: $\|Rp_1 - Rp_2\| = \|p_1 - p_2\|$

Rotations preserve orientation: $(Rp_1) \times (Rp_2) = R(p_1 \times p_2)$

The space of rotations

$$SO(3) = \{ R \in R^{3\times3} \mid RR^T = I, \det(R) = +1 \}$$

Special orthogonal group(3):

Why it's a group:

- Closed under multiplication: if $R_1, R_2 \in SO(3)$ then $R_1 R_2 \in SO(3)$
- Has an identity: $\exists I \in SO(3) \text{ s.t. } IR_1 = R_1$
- Has a unique inverse…
- Is associative…

Why orthogonal:

- vectors in matrix are orthogonal

Why it's special: $\det(R) = +1$, NOT $\det(R) = \pm 1$

Right hand coordinate system

Possible rotation representations

You need at least three numbers to represent an arbitrary rotation in SO(3) (Euler theorem). Some three-number representations:

- ZYZ Euler angles
- ZYX Euler angles (roll, pitch, yaw)
- Axis angle

One four-number representation:

- quaternions

ZYZ Euler Angles

$$r_{zyz} = \begin{pmatrix} \varphi \\ \theta \\ \psi \end{pmatrix}$$

To get from *A* to *B:*

1. Rotate ϕ about *z* axis
2. *Then* rotate θ about y axis
3. *Then* rotate ψ about z axis

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ZYZ Euler Angles

Remember that $R_z(\phi)\ R_y(\theta)\ R_z(\psi)$ encode the desired rotation in the pre-rotation reference frame:

$$R_z(\phi) = {}^{pre-rotation}R_{post-rotation}$$

Therefore, the sequence of rotations is concatentated as follows:

$$R_{zyz}(\phi,\theta,\psi) = R_z(\phi) R_y(\theta) R_z(\psi)$$

$$R_{zyz}(\varphi,\theta,\psi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{zyz}(\varphi,\theta,\psi) = \begin{pmatrix} c_\varphi c_\theta c_\psi - s_\varphi s_\psi & -c_\varphi c_\theta s_\psi - s_\varphi c_\psi & c_\varphi s_\theta \\ s_\varphi c_\theta c_\psi + c_\varphi s_\psi & -s_\varphi c_\theta s_\psi + c_\varphi c_\psi & s_\varphi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{pmatrix}$$

ZYX Euler Angles (roll, pitch, yaw)

To get from *A* to *B:*

1. Rotate ϕ about *z* axis
2. *Then* rotate θ about y axis
3. *Then* rotate ψ about x axis

$$R_z(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$R_y(\theta)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}$$

$$R_x(\psi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sin\psi & \cos\psi\end{pmatrix}$$

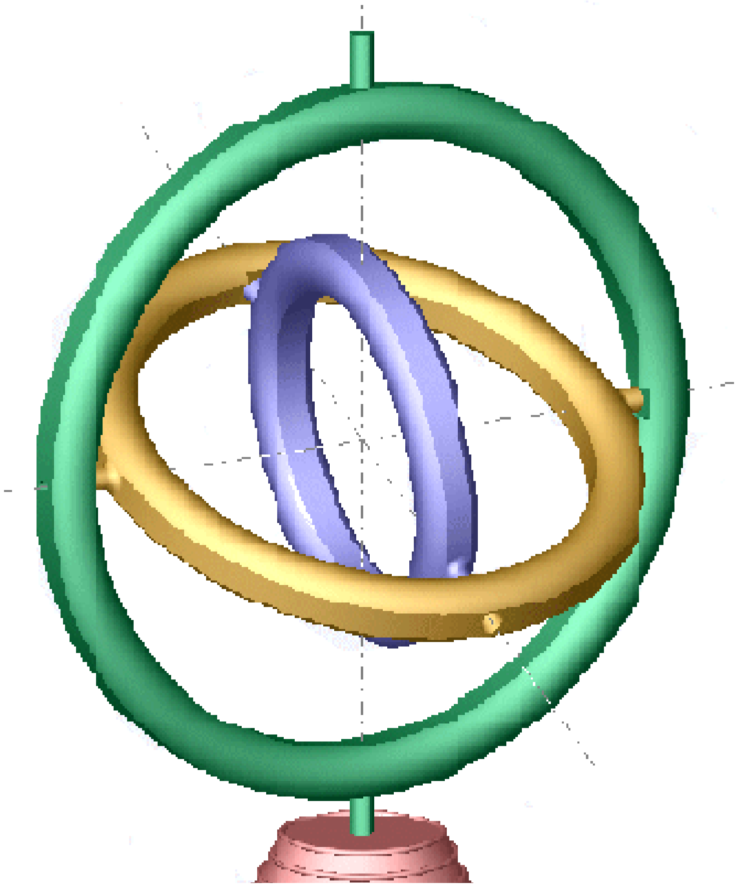
$$R_{zyx}(\phi,\theta,\psi)=R_z(\phi)R_y(\theta)R_x(\psi)$$

$$R_{zyz}(\varphi,\theta,\psi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\psi & -\sin\psi\\ 0 & \sin\psi & \cos\psi\end{pmatrix}$$

Think-pair-share: problems w/ Euler angles

How far apart are these two orientations, actually?

$$r_1 = \begin{pmatrix} 0 \\ 90^\circ \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 90^\circ \\ 89^\circ \\ 90^\circ \end{pmatrix}$$

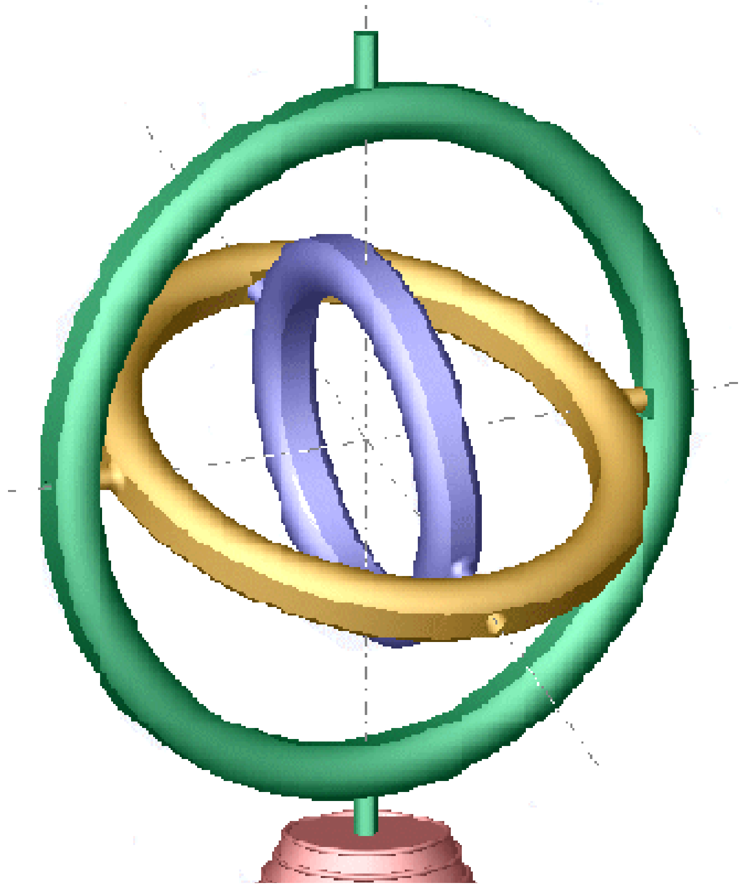


So … differences between Euler angles may not reflect actual distances in orientation

Think-pair-share: problems w/ Euler angles

How far apart are these two orientations, actually?

$$r_1 = \begin{pmatrix} 0 \\ 90^\circ \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 90^\circ \\ 89^\circ \\ 90^\circ \end{pmatrix}$$



So … differences between Euler angles may not reflect actual distances in orientation

An extreme case of this problem is known as "gimbal lock".

- Euler system loses a degree of freedom
- any Euler angle representation can suffer from this

Problem w/ Euler Angles: gimbal lock

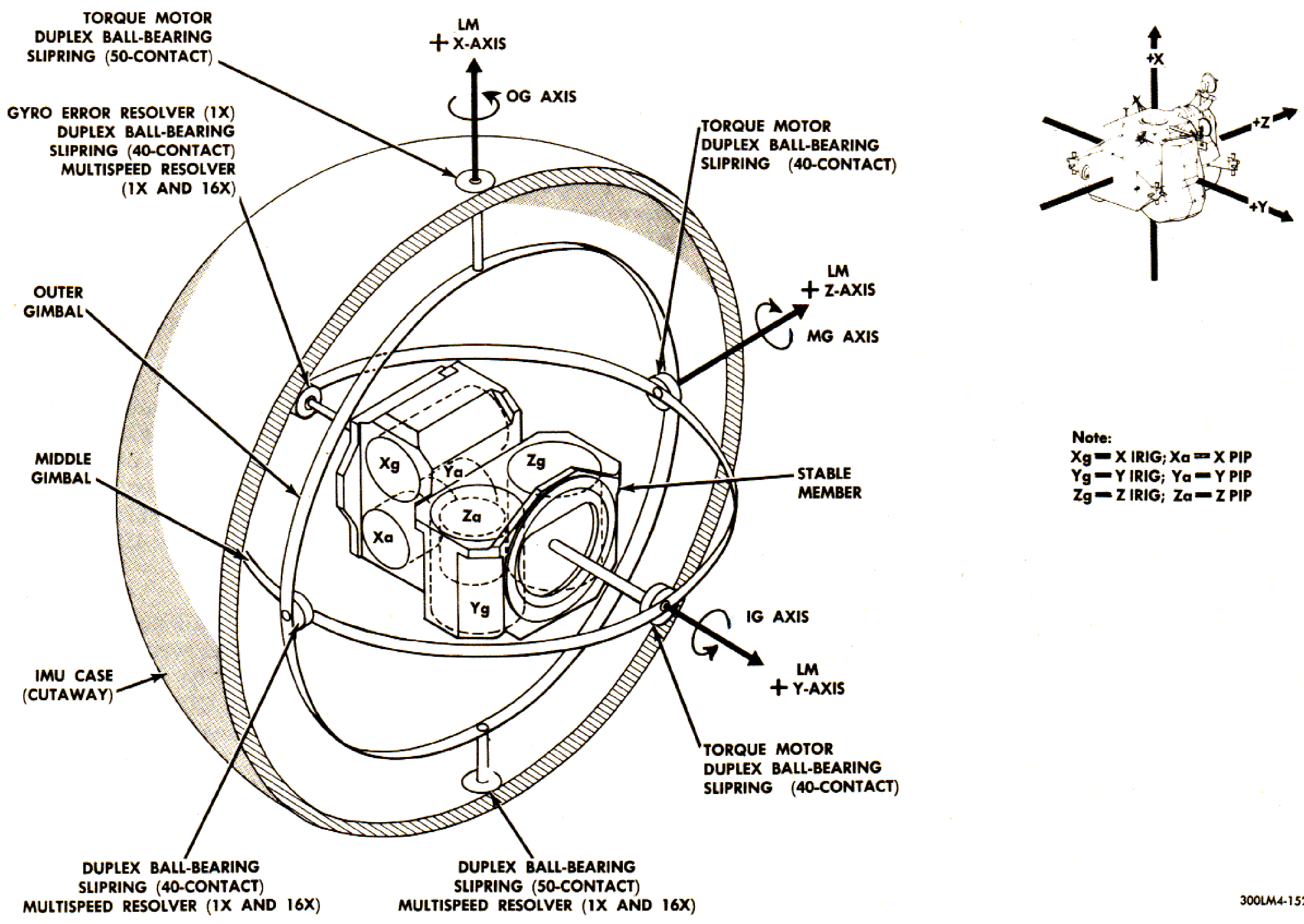
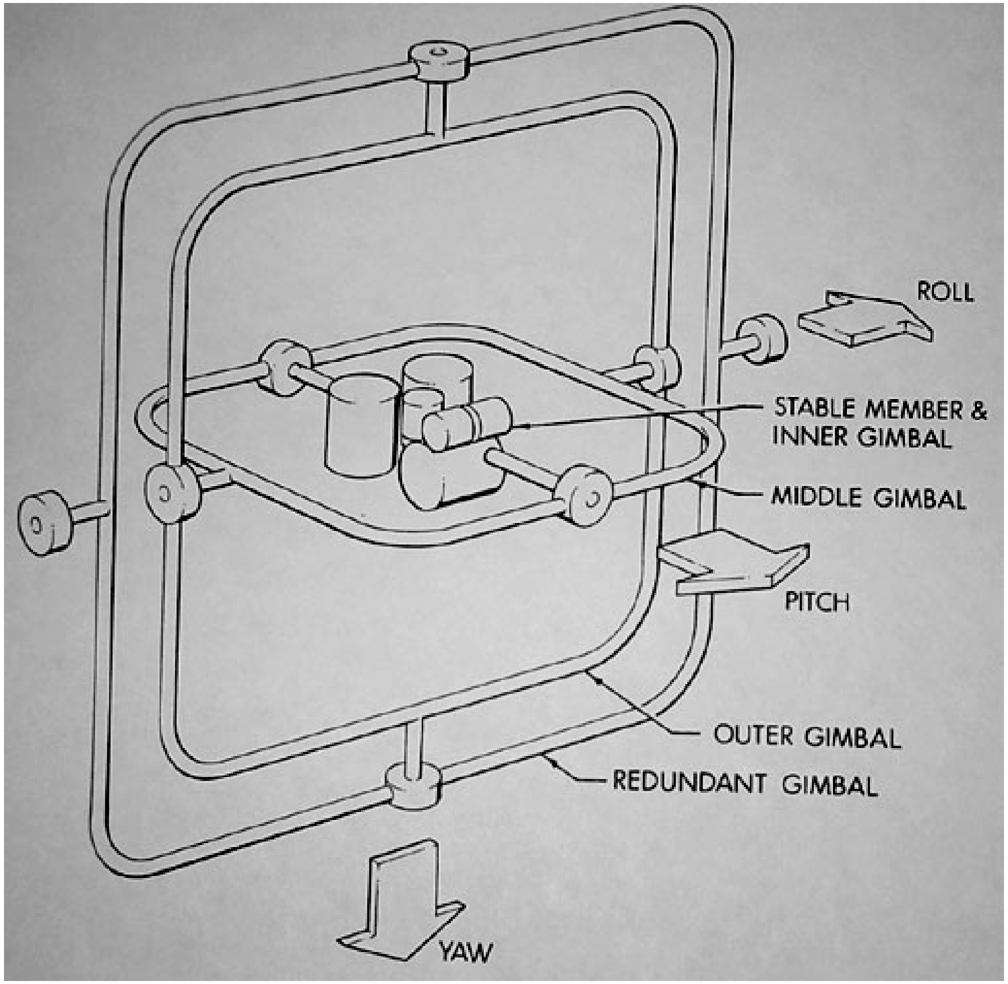


Figure 2.1-24. IMU Gimbal Assembly

Question



Does this solve the problem?

Axis-angle representation

Theorem: (Euler). Any orientation, $R \in SO(3)$, is equivalent to a rotation about a fixed axis, $\omega \in R^3$, through an angle $\theta \in [0, 2\pi)$

(also called *exponential coordinates*)

Axis: $k = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$ Angle: θ

Converting to a rotation matrix:

$$R_{\mathbf{k}\theta} = e^{S(\mathbf{k})\theta} = I + S(\mathbf{k})\sin(\theta) + S(\mathbf{k})^2(1 - \cos(\theta))$$

$$= [\text{that equation in the book...}]$$

Rodrigues' formula

Axis-angle representation

Converting to axis angle:

Magnitude of rotation:

$$\theta = |k| = \cos^{-1}\left(\frac{trace(R) - 1}{2}\right)$$

Axis of rotation:

$$\hat{k} = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

Where:

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{pmatrix}$$

and:

$$trace(R) = r_{11} + r_{22} + r_{33}$$

Axis-angle representation

Axis angle is can be encoded by just three numbers instead of four:

If $k \neq 0$ then $\hat{k} = \frac{k}{|k|}$ and $\theta = |k|$

If the three-number version of axis angle is used, then

$$R_0 = I$$

For most orientations, R_k , is unique.

For rotations of 180°, there are two equivalent representations:

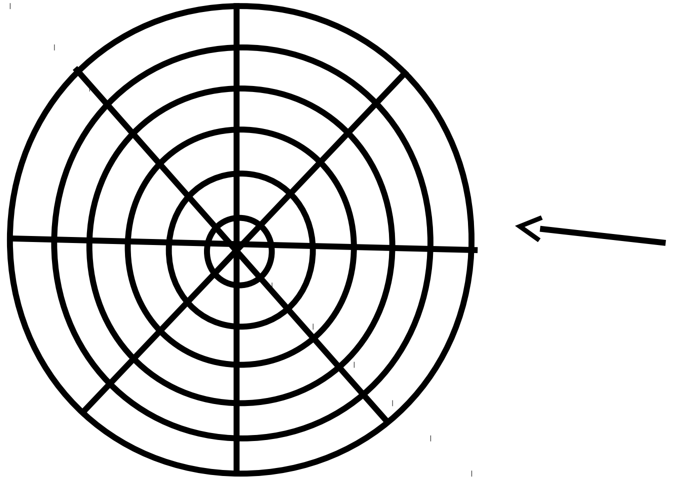
If $|k| = 180^{\circ}$ then $R_k = R_{-k}$

Axis-angle problems

Still suffers from the “edge” and distance preserving problems of Euler angles:

$$r_1 = \begin{pmatrix} 0 \\ 0 \\ 179^\circ \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 \\ 0 \\ -179^\circ \end{pmatrix} \qquad r_1 - r_2 = \begin{pmatrix} 0 \\ 0 \\ 358^\circ \end{pmatrix}$$

, but the actual distance is 2°



Distance metric changes as you get further from origin.

Projection distortions

Example: differencing rotations

Calculate the difference between these two rotations:

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix} \qquad k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

This is NOT the right answer:

$$k_1 - k_2 = \begin{pmatrix} \pi/2 \\ -\pi/2 \\ 0 \end{pmatrix}$$

According to that, this is the magnitude of the difference:

$$\left| k_1 - k_2 \right| = \frac{\pi}{\sqrt{2}} = 127.27^\circ$$

Question

Calculate the difference between these two rotations:

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix} \qquad k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

This is NOT the right answer:

$$k_1 - k_2 = \begin{pmatrix} \pi/2 \\ -\pi/2 \\ 0 \end{pmatrix}$$

According to that, this is the magnitude of the difference:

$$\left| k_1 - k_2 \right| = \frac{\pi}{\sqrt{2}} = 127.27^{\circ}$$

What is the real angular difference between these two orientations?

Quaternions

So far, rotation matrices seem to be the most reliable method of manipulating rotations. But there are problems:

- Over a long series of computations, numerical errors can cause these 3x3 matrices to no longer be orthogonal (you need to “orthogonalize” them from time to time).
- Although you can accurately calculate rotation differences, you can’t interpolate over a difference.’
    - Suppose you wanted to smoothly rotate from one orientation to another – how would you do it?

Answer: quaternions…

Quaternions

Generalization of complex numbers:

$$Q = q_0 + iq_1 + jq_2 + kq_3$$

$$Q = (q_0, q)$$

Essentially a 4-dimensional quantity

Properties of complex dimensions:

$$ii = jj = kk = ijk = -1 \qquad jk = -kj = i$$

$$ij = -ji = k \qquad ki = -ik = j$$

Multiplication:

$$QP = (q_0 + iq_1 + jq_2 + kq_3)(p_0 + ip_1 + jp_2 + kp_3)$$

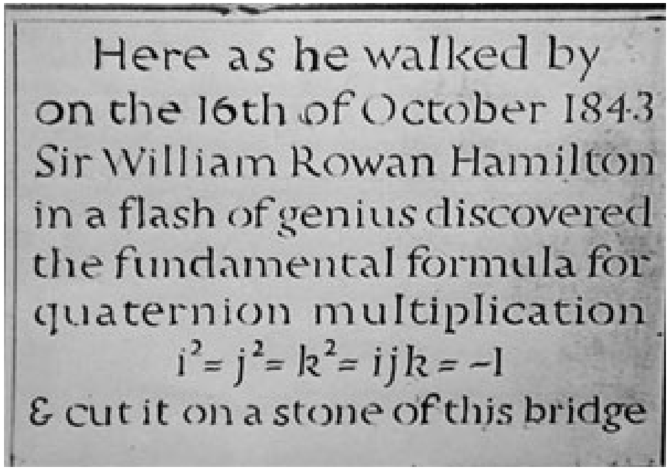
$$QP = (p_0 q_0 - p \cdot q, p_0 q + q_0 p + p \times q)$$

Complex conjugate:

$$Q^* = (q_0, q)^* = (q_0, -q)$$

Quaternions

Invented by Hamilton in 1843:



Along the royal canal in Dublin…

Quaternions

Let's consider the set of unit quaternions: $Q^2 = q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2 = 1$

This is a four-dimensional hypersphere, *i.e.* the 3-sphere S^3

The identity quaternion is: $Q = (1,0)$

Since: $QQ^* = (q_0, q)(q_0, -q) = (q_0 q_0 - q^2, q_0 q - q_0 q + q \times q) = (1,0)$

Therefore, the inverse of a unit quaternion is: $Q^* = Q^{-1}$

Question

Associate a rotation with a unit quaternion as follows:

Given a unit axis, $\hat{k}$, and an angle, θ: ⟵ (just like axis angle)

The associated quaternion is: $$Q_{\hat{k},\theta} = \left(\cos\left(\frac{\theta}{2}\right), \hat{k}\sin\left(\frac{\theta}{2}\right) \right)$$

Therefore, Q represents the same rotation as $-Q$

Question

Associate a rotation with a unit quaternion as follows:

Given a unit axis, $\hat{k}$, and an angle, θ: ← (just like axis angle)

The associated quaternion is: $$Q_{\hat{k},\theta} = \left(\cos\left(\frac{\theta}{2}\right), \hat{k} \sin\left(\frac{\theta}{2}\right) \right)$$

Therefore, Q represents the same rotation as $-Q$

Why?

Quaternions

Associate a rotation with a unit quaternion as follows:

Given a unit axis, $\hat{k}$, and an angle, θ: ← (just like axis angle)

The associated quaternion is: $$Q_{\hat{k},\theta} = \left(\cos\left(\frac{\theta}{2}\right), \hat{k}\sin\left(\frac{\theta}{2}\right) \right)$$

Therefore, Q represents the same rotation as $-Q$

Let ${}^{i}P = \left(0, {}^{i}p\right)$ be the quaternion associated with the vector ${}^{i}p$

You can rotate ${}^{a}P$ from frame *a* to *b*: ${}^{b}P = Q_{ba}\,{}^{a}P\,Q_{ba}^{\;*}$

Composition: $Q_{ca} = Q_{cb} Q_{ba}$

Inversion: $Q_{cb} = Q_{ca} Q_{ba}^{\;-1}$

Example: Quaternions

Rotate $^{a}P = \left(0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ by $Q = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)$

$$^{b}P = Q\,^{a}PQ^{¿} = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)\left(0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)$$

$$= \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)\left(0, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right)$$

$$= \left(0, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right)$$

Example: Quaternions

Find the difference between these two axis angle rotations:

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix} \qquad k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad Q_{cb} = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right) \qquad Q_{ba} = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}\right)$$

$$QP = \left(p_0 q_0 - p \cdot q \, , p_0 q + q_0 p + p \times q\right)$$

$$Q_{cb} = Q_{ca} Q_{ba^{-1}} = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}\right)$$

$$= \left(\frac{1}{2}, \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right) = \left(\frac{1}{2}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}\right)$$

$$\theta_{cb} = \cos^{-1}\left(\frac{1}{2}\right) = \frac{2}{3}\pi$$

$$k_{cb} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

Quaternions: Interpolation

Suppose you're given two rotations, R_1 and R_2

How do you calculate intermediate rotations?

$$R_i = \alpha R_1 + (1-\alpha) R_2$$

← This does not even result in a rotation matrix

Do quaternions help?

$$Q_i = \frac{\alpha Q_1 + (1-\alpha) Q_2}{\left|\alpha Q_1 + (1-\alpha) Q_2\right|}$$

Suprisingly, this actually works

- Finds a geodesic

This method normalizes automatically (SLERP):

$$Q_i = \frac{Q_1 \sin(1-\alpha)\Omega + Q_2 \sin \alpha\Omega}{\sin \Omega}$$