Propositional Logic

Logic is the study of reasoning and sound arguments. At its simplest, logic is what you use to perform the following kind of reasoning. Suppose that whenever it rains it is cloudy. It is raining right now. Therefore, it is cloudy. For a slightly more complex example, suppose that whenever it rains it is cloudy, and whenever it is cloudy then there is less light. It is raining right now, therefore there is less light right now. For a more involved but still elementary example, we know that if you are at least 21 years old you can buy alcohol legally in MA. Suppose I know that you cannot buy alcohol legally in MA. Then I know that you must be less than 21 years old.

The feature of the above arguments is that they are all about sentences ("it is raining", "it is cloudy", "you are at least 21 years old", etc) linked together with *logical connectives*, such as "and", "or", "implies" (or "if...then", "whenever", "therefore"). Propositional logic (also known as Boolean logic) is the study of those logical connectives.

Let us formalize propositional logic with a simple language. A formula of propositional logic (written φ or ψ) is one of the following:

- $\mathit{true}, \mathit{false}$ (constants)
- $p, q, r, s, \ldots$ (propositional variables)
- $\neg\varphi$ (negation)
- $\varphi \wedge \psi$ (conjunction)
- $\varphi \vee \psi$ (disjunction)
- $\varphi \Rightarrow \psi$ (implication)
- $\varphi \equiv \psi$ (equivalence, also written $\varphi \Leftrightarrow \psi$)).

This lets you build complex formulas, such as $((p \Rightarrow q) \wedge p) \Rightarrow q$—which you can think of as the representation in propositional logic of the "it is raining" example above, if you read p as "it is raining" and q as "it is cloudy". (More on how to convert English into propositional logic later.)

All we have defined is a syntax—a way to write down formulas. We haven't said how to *interpret* those formulas. Intuitively, formulas are true or false, depending on the truth or falsity of the propositional variables they contain.

Clearly, constant *true* has truth value T, and constant *false* has truth value F. (I'm being somewhat pedantic in distinguishing the constant *true*, which is syntax, from the truth value T; we'll ditch the distinction soon enough.) The meaning of the logical connectives is given by *truth tables*, which I hope dearly you have seen at some point in the past.

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

Two things to note. First, the interpretation of $\vee$ is slightly different than that of common English usage, which takes "A or B" to mean either A or B but not both—as in "Hands up! Your money or your life!"—whereas $p \vee q$ is true if either p or q or both are true. Second, the interpretation of $p \Rightarrow q$ is somewhat nonintuitive. In particular, if the premise p is false, then $p \Rightarrow q$ is true irrespectively of what q is. You just have to accept it—it's the most reasonable choice, it turns out.

Once you have the above truth table, you can devise a truth table for any formula φ by combining the truth tables for all the subformulas of φ. For instance, consider our example above.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(1)

Definition 1 *A* truth assignment *for φ is an assignment of truth values to each of the propositional variables of a formula φ. A truth assignment for φ is called a* satisfying assignment *for φ if it makes formula φ true.*

Definition 2 *A formula is* satisfiable *if it has a satisfying assignment. A formula is* valid *if every truth assignment is a satisfying assignment. (A valid formula is sometimes called a* tautology.*)*

Put differently, a formula is valid if it is true no matter what we take to be the truth value of its propositional variables.

How do you check validity of a formula? The most basic way is simply to compute the truth table of the formula and make sure that all truth assignments make the formula true. Truth table (1) shows that $((p \Rightarrow q) \wedge p) \Rightarrow q$ is a valid formula.

Fact 3 *If φ is valid, then φ is satisfiable. (Why?)*

Fact 4 *Formula φ is valid exactly when $\neg\varphi$ is not satisfiable. (Why?)*

Exercise 5 *Check that the following formulas are valid:*

(1) $\neg\neg p \equiv p$

(2) $(p \wedge q) \Rightarrow p$

(3) $(p \wedge q) \Rightarrow q$

(4) $p \wedge q \equiv q \wedge p$

(5) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(6) $p \Rightarrow (p \vee q)$

(7) $q \Rightarrow (p \vee q)$

(8) $p \vee q \equiv q \vee p$

(9) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(10) $(p \equiv q) \equiv (q \equiv p)$

(11) $(p \equiv q) \wedge (q \equiv r) \Rightarrow (p \equiv r)$

(12) $(p \equiv q) \Rightarrow (p \Rightarrow q)$

(13) $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$

Once we have a bank of valid formulas, then we can derive new valid formulas without necessarily constructing a truth table. Here is one way to do it.

Principle of Specialization: If a formulas φ is valid, replacing every occurrence of a propositional variable in φ by (the same) formula yields a valid formula.

Thus, since we know that $p \wedge q \equiv q \wedge p$, by replacing p by $(r \Rightarrow s) \vee t$, we immediately get that $((r \Rightarrow s) \vee t) \wedge q \equiv q \wedge ((r \Rightarrow s) \vee t)$ is valid. (Check if you have any doubt, by constructing a truth table.)

Note that the Principle of Specialization only works for validity, not for satisfiability. It is *not necessarily the case* that if φ is satisfiable we can replace all occurrences of a propositional variable by some formula and still obtain a satisfiable formula. (Why?—warning, not easy.)

Definition 6 *Two formulas φ and ψ are* equivalent *if they have the same truth value for every truth assignment; put differently, φ and ψ are equivalent exactly when $\varphi \equiv \psi$ is valid.*

Formula equivalence tells you that two formulas mean the same thing. We now use equivalence to show that some logical connectives are interdefinable.

For instance, you can express $\wedge$ in terms of $\vee$ and $\neg$. The following formula is valid:

$$p \wedge q \equiv \neg(\neg p \vee \neg q)$$

Check it by building a truth table. By the Principle of Specialization, you can then obtain equivalence for any conjunction of formulas.

Similarly, you can express $\vee$ in terms of $\wedge$ and $\neg$. The following formula is valid:

$$p \vee q \equiv \neg(\neg p \wedge \neg q) \tag{2}$$

Again, check it! Implication can be written in terms of $\vee$ and $\neg$. The following formula is valid:

$$p \Rightarrow q \equiv (\neg p) \vee q \tag{3}$$

Again, check it! Finally, equivalence can be written in terms of $\wedge$ and $\Rightarrow$; the following formula is valid:

$$(p \equiv q) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$$

To rewrite arbitrary formulas using the above equivalences, we need a new principle.

Principle of Substitution: If $\psi_1 \equiv \psi_2$, then φ is valid if and only if φ where we replace an occurrence of ψ_1 by ψ_2 is valid.

We can use the Principle of Substitution and that of Specialization to show that $p \Rightarrow q$ can be written using only $\wedge$ and $\neg$:

> Start with $p \Rightarrow q$. By equivalence (3) above, we know that $p \Rightarrow q$ is equivalent to $(\neg p) \vee q$. Equivalence (2) tells us that $p \vee q \equiv \neg(\neg p \wedge \neg q)$ is valid. The Principle of Specialization tells us if we replace p by $\neg p$, we get a valid formula
>
> $$(\neg p) \vee q \equiv \neg(\neg\neg p \wedge \neg q). \tag{4}$$
>
> Now, Exercise 5 tells us that $\neg\neg p \equiv p$ is valid. Thus, we can use the Principle of Substitution to replace $\neg\neg p$ by p in (4), and get that $(\neg p) \vee q \equiv \neg(p \wedge \neg q)$, that is, $(\neg p) \vee q$ is equivalent to $\neg(p \wedge \neg q)$. Putting it all together, we have that $p \Rightarrow q$ is equivalent to $(\neg p) \vee q$ is equivalent to $\neg(p \wedge \neg q)$, and so $p \Rightarrow q \equiv \neg(p \wedge \neg q)$ is valid.

Okay, so this is a bit long. It may actually be faster to come up with the truth table for the final equivalence to check that it is a validity. But if you understand the above, you can transform valid formulas into other valid formulas pretty quickly. We will make this kind of reasoning precise next lecture.

More importantly, the argument above can be generalized to show that no matter what formula you give me, I can rewrite all its $\equiv$ into $\wedge$ and $\Rightarrow$, and take the result and rewrite all $\Rightarrow$ into $\vee$ and $\neg$, and take the result and rewrite all $\vee$ into $\wedge$ and $\neg$, and I end up with a formula written only in terms of $\wedge$ and $\neg$ (and possibly the constants *true* and *false*) that is equivalent to the original formula.

Can we do better? Can we reduce to a single connective? It turns out that yes, but we need to introduce it. All the logical connectives can be written using a single connective, **if** φ **then** ψ_1 **else** ψ_2, with following truth table:

p	q	r	**if** p **then** q **else** r
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

(We sometimes write **if** φ **then** ψ_1 **else** ψ_2 as $ite(\varphi, \psi_1, \psi_2)$.)

Exercise 7 *Show the following formulas are valid:*

(1) $p \wedge q \equiv$ **if** p **then** q **else** *false*

(2) $\neg p \equiv$ **if** p **then** *false* **else** *true*

Exercise 8 *Can you come up with a formula using only* **if** – **then** – **else** – *(as well as true and false) that is equivalent to the following formulas:*

(1) $p \vee q$

(2) $p \Rightarrow q$

(3) $p \equiv q$

Show that your answers are in fact equivalences.

Exercise 9 *If this is too easy for you, here is something to keep you distracted.* **if** $-$ **then** $-$ **else** $-$ *is a three-arguments operation. It turns out that a single binary operation can still express all the other logical connective. The operation is written* $p \mid q$*, with the following truth table:*

p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

Come up with equivalences for $p \wedge q$*,* $\neg p$*,* $p \vee q$*,* $p \Rightarrow q$ *and* $p \equiv q$*, all written using only the* $- \mid -$ *operation (and possibly constants* true *and* false*).*