Classical Cryptography

CS 6750 Lecture 1

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Riccardo Pucella

Goals of Classical Cryptography

- Alice wants to send message X to Bob
- Oscar is on the wire, listening to all communications
- Alice and Bob share a secret K
- Alice converts X into Y using secret K
- Alice sends Y to Bob
- Bob converts Y back to X using secret K

- Goal: protect message X from Oscar
    - Better goal: protect secret K from Oscar

Shift Cipher

- Earliest example of a cryptosystem
- Given a string M of letters
  - For simplicity, assume only capital letters of English
  - Remove spaces
- Key k: a number between 0 and 25
- To encrypt, replace every letter by the one k places down the alphabet (wrapping around)
- To decrypt, replace every letter by the one k places up the alphabet (wrapping around)
- Example: k=10, THISISSTUPID → DRSCSCCDEZSN

Definition of Cryptosystem

- A cryptosystem is a tuple (P,C,K,E,D) such that:
  1. P is a finite set of possible plaintexts
  2. C is a finite set of possible ciphertexts
  3. K is a finite set of possible keys (keyspace)
  4. For every k, there is an encryption function $e_k \in E$ and decryption function $d_k \in D$ such that $d_k(e_k(x)) = x$ for all plaintexts x.
- Encryption function assumed to be injective
- Encrypting a message:

$$x = x_1\ x_2\ \dots\ x_n \quad \rightarrow \quad e_k(x) = e_k(x_1)\ e_k(x_2)\ \dots\ e_k(x_n)$$

Properties of Cryptosystems

- Encryption and decryption functions can be efficiently computed
- Given a ciphertext, it should be difficult for an opponent to identify the encryption key and the plaintext
- For the last to hold, the key space must be large enough!
  - Otherwise, may be able to iterate through all keys

Shift Cipher, Revisited

- $P = Z_{26} = \{0,1,2,...,25\}$
  - Encoding: A = 0, B = 1, ..., Z = 25
- $C = Z_{26}$
- $K = Z_{26}$
- $e_k = ?$
  - Add k, and wraparound...

Modular Arithmetic

- Congruence
  - a, b: integers   m: positive integer
  - $a \equiv b \pmod{m}$ iff m divides $a-b$
    - a congruent to b modulo m
  - Examples: $75 \equiv 11 \pmod{8}$ $75 \equiv 3 \pmod{8}$
  - Given m, every integer a is congruent to a unique integer in $\{0,\ldots,m-1\}$
    - Written a (mod m)
    - Remainder of a divided by m

Modular Arithmetic

- $Z_m = \{ 0, 1, \ldots, m-1 \}$
- Define a + b in Z_m to be a + b (mod m)
- Define a x b in Z_m to be a x b (mod m)
- Obeys most rules of arithmetic
  - + commutative, associative, 0 additive identity
  - x commutative, associative, 1 mult. identity
  - + distributes over x
  - Formally, Z_m forms a ring
    - For a prime p, Z_p is actually a field

Shift Cipher, Formally

- $P = Z_{26} = \{0,1,2,...,25\}$ (where A=0, B=1,..., Z=25)
- $C = Z_{26}$
- $K = Z_{26}$
- $e_k(x) = x + k \pmod{26}$
- $d_k(y) = y - k \pmod{26}$
- Size of the keyspace? Is this enough?

Affine Cipher

- Let's complicate the encryption function a little bit
  - $K = Z_{26} \times Z_{26}$ (tentatively)
  - $e_k(x) = (ax + b) \bmod 26$, where $k=(a,b)$

- How do you decrypt?
  - Given a,b, and y, can you find $x \in Z_{26}$ such that

$$(ax+b) \equiv y \pmod{26}?$$

or equivalently: $ax \equiv y-b \pmod{26}$?

Affine Cipher

Theorem: $ax \equiv y \pmod{m}$ has a unique solution $x \in Z_m$ iff $\gcd(a,m)=1$

- In order to decrypt, need to find a unique solution
  - Must choose only keys (a,b) such that $\gcd(a,26)=1$
- Let a^{-1} be the solution of $ax = 1 \pmod{m}$
  - Then $a^{-1}b$ is the solution of $ax = b \pmod{m}$

Affine Cipher, Formally

- $P = C = Z_{26}$
- $K = \{ (a,b) \mid a,b \in Z_{26}, \gcd(a,26)=1 \}$
- $e_{(a,b)}(x) = ax + b \pmod{26}$
- $d_{(a,b)}(y) = ?$

- What is the size of the keyspace?
  - (Number of a's with gcd(a,26)=1) x 26
  - $\varphi(26) \times 26$

Substitution Cipher

- $P = Z_{26}$
- $C = Z_{26}$
- K = all possible permutations of Z_{26}
  - A permutation P is a bijection from Z_{26} to Z_{26}
- $e_k(x) = k(x)$
- $d_k(x) = k^{-1}(x)$
  - Example
  - Shift cipher, affine cipher
- Size of keyspace?

Cryptanalysis

- Kerckhoff's Principle:
  - The opponent knows the cryptosystem being used
  - No "security through obscurity"
- Objective of an attacker
  - Identify secret key used to encrypt a ciphertext
- Different models of attackers to consider:
  - Ciphertext only attack
  - Known plaintext attack
  - Chosen plaintext attack
  - Chosen ciphertext attack

Cryptanalysis of Substitution Cipher

- Statistical cryptanalysis
  - Ciphertext only attack
- Again, assume plaintext is English, only letters
- Goal of the attacker: determine the substitution
- Idea: use statistical properties of English text

Statistical Properties of English

- Letter probabilities (Beker and Piper, 1982): p_0, ..., p_{25}
- A: 0.082, B: 0.015, C: 0.028, ...
- More useful: ordered by probabilities:
  - E: 0.120
  - T,A,O,I,N,S,H,R: [0.06, 0.09]
  - D,L: 0.04
  - C,U,M,W,F,G,Y,P,B: [0.015, 0.028]
  - V,K,J,X,Q,Z: < 0.01
- Most common digrams: TH,HE,IN,ER,AN,RE,ED,ON,ES,ST...
- Most common trigrams: THE,ING,AND,HER,ERE,ENT,...

Statistical Cryptanalysis

General recipe:

- Identify possible encryptions of E (most common English letter)
  - T,A,O,I,N,S,H,R: probably difficult to differentiate
- Identify possible digrams starting/finishing with E (-E and E-)
- Use trigrams
  - Find 'THE'
- Identify word boundaries

Polyalphabetic Ciphers

- Previous ciphers were monoalphabetic
  - Each alphabetic character mapped to a unique alphabetic character
  - This makes statistical analysis easier
- Obvious idea
  - Polyalphabetic ciphers
  - Encrypt multiple characters at a time

Vigenère Cipher

- Let m be a positive integer (the key length)
- $P = C = K = Z_{26} \times \ldots \times Z_{26} = (Z_{26})^m$
- For $k = (k_1, \ldots, k_m)$:
  - $e_k(x_1, \ldots, x_m) = (x_1 + k_1 \pmod{26}, \ldots, x_m + k_m \pmod{m})$
  - $d_k(y_1, \ldots, y_m) = (y_1 - k_1 \pmod{26}, \ldots, y_m - k_m \pmod{m})$
- Size of keyspace?

Cryptanalysis of Vigenère Cipher

- Thought to thwart statistical analysis, until mid-1800
- Main idea: first figure out key length (m)
  - Two identical segments of plaintext are encrypted to the same ciphertext if they are δ position apart, where $\delta = 0 \pmod{m}$
  - Kasiski Test: find all identical segments of length > 3 and record the distance between them: δ_1, δ_2, ...
    - m divides $\gcd(\delta_1, \delta_2, \ldots)$

Index of Coincidence

- We can get further evidence for the value of m as follows
- The index of coincidence of a string $X = x_1 \ldots x_n$ is the probability that two random elements of X are identical
  - Written $I_c(X)$
- Let f_i be the # of occurrences of letter i in X; $I_c(X)$ = ?
- For an arbitrary string of English text, $I_c(X) \approx 0.065$
  - If X is a shift ciphertext from English, $I_c(X) \approx 0.065$
- For m=1,2,3,... decompose ciphertext into substrings y_i of all m^{th} letters; compute I_c of all substrings
  - I_cs will be ≈ 0.065 for the right m
  - I_cs will be ≈ 0.038 for wrong m

Then what?

- Once you have a guess for m, how do you get keys?
- Each substring y_i:
    - Has length $n' = n/m$
    - Encrypted by a shift k_i
    - Probability distribution of letters: $f_0/n', \ldots, f_{25}/n'$
- $f_{0+ki \pmod{26}}/n', \ldots, f_{25+ki \pmod{26}}/n'$ should be close to $p_0, \ldots, p_{25}$
- Let $M_g = \sum_{i=0,\ldots,25} p_i \, (f_{i+g \pmod{26}} / n')$
    - If $g = k_i$, then $M_g \approx 0.065$
    - If $g \neq k_i$, then M_g is usually smaller

Hill Cipher

- A more complex form of polyalphabetic cipher
- Again, let m be a positive integer
- $P = C = (Z_{26})^m$
- To encrypt: (case m=2)
  - Take linear combinations of plaintext (x_1, x_2)
  - E.g., $y_1 = 11\ x_1 + 3\ x_2 \pmod{26}$
    $y_2 = 8\ x_1 + 7\ x_2 \pmod{26}$
  - Can be written as a matrix multiplication (mod 26)

Hill Cipher, Continued

- $K = Mat(Z_{26}, m)$ (tentatively)
- $e_k(x_1, \ldots, x_m) = (x_1, \ldots, x_m)\, k$
- $d_k(y_1, \ldots, y_m) = ?$
  - Similar problem as for affine ciphers
  - Want to be able to reconstruct plaintext
  - Solve m linear equations (mod 26)
  - I.e., find k^{-1} such that kk^{-1} is the identity matrix
    - Need a key k to have an inverse matrix k^{-1}

Cryptanalysis of Hill Cipher

- Much harder to break with ciphertext only
- Easy with known plaintext
- Recall: want to find secret matrix k
- Assumptions:
  - m is known
  - Construct m distinct plaintext-ciphertext pairs
    - $(X_1, Y_1), \ldots, (X_m, Y_m)$
- Define matrix Y with rows $Y_1, \ldots, Y_m$
- Define matrix X with rows $X_1, \ldots, X_m$
- Verify: $Y = X\,k$
- If X is invertible, then $k = X^{-1}\,Y$!