The Mathematics of Game Show Scheduling

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1 Introduction

In August 2010, I received the following e-mail from the Executive Producer of a television show:

> *Hi Richard,*
>
> *I'm working on an action/adventure game show for YTV called "Splatalot" and I need your help. The game sees kids competing on an giant obstacle course. We have 9 Defenders of the castle – 3 from Canada, 3 from the UK and 3 from Australia. They are actors hired by our production to "defend the castle". Only 2 from each territory ever appear in an episode. So 9 Defenders but only 6 appear in any one episode. I'm wondering if you can help us schedule our Defenders?*
>
> *We have 26 episodes. Each episode has a different set of 6 Defenders. Each episode must have 2 Defenders from each territory. We'd like it so that each performer shoots 3 or 4 episodes then has a day off (or as close as we can get to this type of schedule).*
>
> *Am I making sense? Please let me know if you have any questions. Sorry for the cold call, but we just can't seem to work it out. Thanks in advance. Hope you have been keeping well!*

Splatalot premiered in March 2011 on YTV (Canada), BBC (Great Britain), and ABC (Australia). In each episode, twelve teenagers competed as "attackers", racing against the clock to complete the medieval-themed obstacle course in their quest to be crowned the King or Queen of *Splatalot*. The defenders' role was to slow down the contestants and protect the castle, ensuring numerous spills and "splats". The show was a big hit in Canada, especially among high school students.

In this note, we present our graph-theoretic solution to the TV producer's scheduling optimization problem, and conclude with a challenge tying concepts from Ramsey Theory and Design Theory for the generalized scenario of c countries, d defenders per country, and e defenders per episode.

2 Solving the Problem

The nine defenders were Australian, British, and Canadian, with three from each country. We begin by labeling them $\{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$.

Each episode consists of six defenders, with two from each country. As a result, there are $\binom{3}{2}^3 = 27$ possible ways the defenders can be selected. As the producer only needed 26 episodes, each show's taping could consist of a unique subset of six defenders. To give an example of a complete 26-episode schedule, suppose we arrange the 27 possible episodes *lexicographically*, ignoring the final column. This is illustrated in Table 1, where each entry is marked is binary, with 1 representing a "play" and 0 representing a "rest". By definition, each column vector is unique.

Given any schedule Γ, corresponding to a permutation of the 27 possible column vectors, let $p := p(\Gamma)$ be the maximum number of consecutive episodes played by any defender, and let $r := r(\Gamma)$ be the maximum number of consecutive episodes rested by any defender. In Table 1, $p = 18$ and $r = 9$, due to the schedules of defenders A_1 and A_3. Clearly $p \leq 18$ and $r \leq 9$, which makes Table 1 the worst possible schedule that could be chosen among all 27! options.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**
A_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
A_2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
A_3	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B_1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0
B_2	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1
B_3	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
C_1	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
C_2	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
C_3	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1

Table 1. The lexicographic ordering of the 27 episodes.

The TV producer wanted a schedule Γ that minimized the values of $p := p(\Gamma)$ and $r := r(\Gamma)$, in order to reduce defender fatigue and boredom. We will show that $p \geq 3$ and $r \geq 1$, and generate a schedule Γ for which these optimal values are attained, i.e., in the corresponding 9×27 matrix, no row contains a 1111-substring or a 00-substring. To do that, we apply the following result, whose proof is straightforward.

Proposition 1 *Suppose there are c countries, d defenders per country, and e defenders per episode. Let Γ be a schedule, corresponding to a permutation of the $\binom{d}{e}^c$ possible column vectors. Let $p := p(\Gamma)$ be the maximum number of consecutive episodes played by any defender, and let $r := r(\Gamma)$ be the maximum number of consecutive episodes rested by any defender. Then $p > \frac{e}{d-e}$ and $r > \frac{d-e}{e}$.*

In the case of *Splatalot*, we have $(c, d, e) = (3, 3, 2)$, and so $p \geq 3$ and $r \geq 1$. We now create a schedule with $(p, r) = (3, 1)$. To do this, consider the set of $\binom{d}{e}^c = 27$ *hyperedges* that contain six elements from $\{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$, with exactly two from each country. Label these 27 hyperedges $\{e_1, e_2, \ldots, e_{27}\}$ in lexicographic order (see Table 1). For example, $e_1 = \{A_1, A_2, B_1, B_2, C_1, C_2\}$ and $e_{27} = \{A_2, A_3, B_2, B_3, C_2, C_3\}$.

Construct a graph G with vertices $\{e_1, e_2, \ldots, e_{27}\}$, where vertices e_i and e_j are adjacent iff $e_i \cup e_j = \{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$. It is easy to see that G contains 108 edges, since each of the 27 vertices has $2 \times 2 \times 2 = 8$ neighbours.

Consider any *Hamiltonian path* of G, i.e., a 26-edge path covering all the vertices in the order $H_1, H_2, \ldots, H_{27}$. By definition, the set $\{H_1, H_2, \ldots, H_{27}\}$ is a permutation of $\{e_1, e_2, \ldots, e_{27}\}$. By setting H_i to be the i^{th} episode of our schedule, we produce a schedule with $r = 1$, since for every $1 \leq j \leq 26$, each defender appears at least once in $H_j \cup H_{j+1} = \{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$.

To also ensure $p = 3$, we require that no defender appears in four consecutive episodes, i.e., $\overline{H_j} \cup \overline{H_{j+1}} \cup \overline{H_{j+2}} \cup \overline{H_{j+3}} = \{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$ for each $1 \leq j \leq 24$. A simple computer search finds multiple solutions, including the following Hamiltonian path:

$$
\begin{array}{ccccccccccccccccc}
e_{27} & \rightarrow & e_{14} & \rightarrow & e_7 & \rightarrow & e_{20} & \rightarrow & e_{15} & \rightarrow & e_8 & \rightarrow & e_{22} & \rightarrow & e_{12} & \rightarrow & e_5 \\
 & & & & & & & & & & & & & & & & \downarrow \\
e_2 & \leftarrow & e_{16} & \leftarrow & e_{21} & \leftarrow & e_4 & \leftarrow & e_{17} & \leftarrow & e_{19} & \leftarrow & e_6 & \leftarrow & e_{11} & \leftarrow & e_{25} \\
\downarrow & & & & & & & & & & & & & & & & \\
e_{24} & \rightarrow & e_{10} & \rightarrow & e_9 & \rightarrow & e_{23} & \rightarrow & e_{18} & \rightarrow & e_1 & \rightarrow & e_{26} & \rightarrow & e_{13} & \rightarrow & e_3
\end{array}
$$

This Hamiltonian path gives us a numbering of the 27 hyperedges corresponding to the episode order. This produces a valid schedule Γ solving the *Splatalot* problem, with $p = 3$ and $r = 1$. One can quickly verify that Table 2 has 27 unique columns, and no row contains either a 00- or 1111-substring. This was the final schedule sent to the Executive Producer.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**
A_1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
A_2	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
A_3	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
B_1	0	1	0	1	1	0	1	1	1	0	1	1	1	0	1	1	0	1	1	1	0	1	0	1	0	1	1
B_2	1	0	1	1	0	1	0	1	0	1	1	0	1	1	0	1	1	1	0	1	1	0	1	1	1	0	1
B_3	1	1	1	0	1	1	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	1	1	0	1	1	0
C_1	0	1	1	1	0	1	1	0	1	1	1	0	1	1	1	0	1	1	0	1	0	1	0	1	1	1	0
C_2	1	0	1	0	1	0	1	1	0	1	0	1	1	0	1	1	1	0	1	1	1	0	1	1	0	1	1
C_3	1	1	0	1	1	1	0	1	1	0	1	1	0	1	0	1	0	1	1	0	1	1	1	0	1	0	1

Table 2. A solution to the *Splatalot* scheduling problem.

3 Conclusion

Table 2 is a valid solution for the triplet $(c, d, e) = (3, 3, 2)$: each column consists of a unique subset, with e out of d defenders chosen from each of the c countries; no defender plays more than $p = 3$ consecutive games; and no defender rests more than $r = 1$ consecutive games.

However, could this schedule be "optimized" even further? In Table 2, note that B_1 plays six times between Episodes 7 and 13, but only twice between Episodes 21 and 25. And so, there is an imbalance as A_1 plays at most five times during any stretch of seven consecutive episodes, and at least three times during any stretch of five consecutive episodes. It would be interesting to develop further criteria to determine whether a schedule is "optimally balanced" to find the best possible solution for the case $(c, d, e) = (3, 3, 2)$.

This problem has a natural connection to a topic in Design Theory known as *interval-balanced tournament designs* [1], where the appearances of each element are equitably distributed so that each individual rests at least some minimum amount of rounds between its matches and that this minimum rest is maximized. However, our problem appears even harder as it requires two parameters, p and r, to be optimized simultaneously. This motivates the following two-parameter problem formulated in the language of Ramsey Theory:

Problem 1 *Suppose there are c countries, d defenders per country, and e defenders per episode. Determine the largest values of $p := p(c, d, e)$ and $r := r(c, d, e)$ for which the following statement is true: For any schedule Γ, corresponding to a permutation of the $\binom{d}{e}^c$ possible column vectors, at least one defender plays p consecutive games, and at least one defender rests r consecutive games.*

We conjecture that $p = \lfloor \frac{e}{d-e} \rfloor + 1$ and $r = \lfloor \frac{d-e}{e} \rfloor + 1$, but have only verified this for small triplets (c, d, e). While we were able to solve the TV producer's conundrum and give him an optimal schedule for *Splatalot*, we have only scratched the surface. The problems in this section suggest fascinating and deep explorations into areas of Design Theory and Ramsey Theory, and we leave these as challenges for the interested reader.

References

1. Rodney, P. (1993). Balance in tournament designs. Ph.D. thesis, University of Toronto.