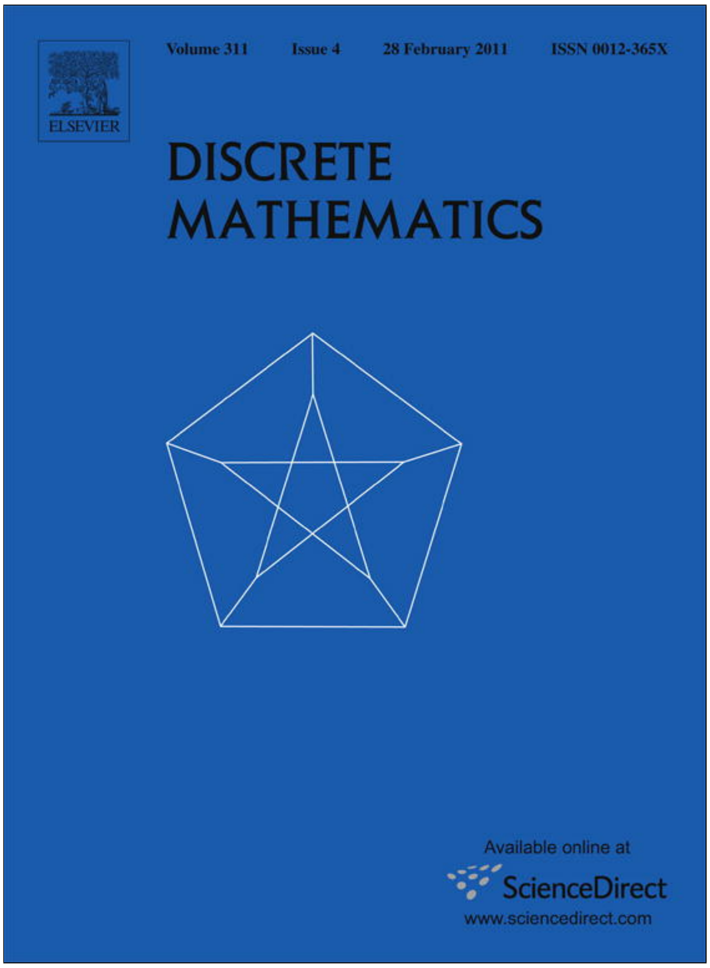
ELSEVIER
Volume 311 Issue 4 28 February 2011 ISSN 0012-365X
DISCRETE
MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 311 (2011) 244–251

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Well-covered circulant graphs

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ARTICLE INFO

Article history:
Received 11 June 2010
Received in revised form 8 November 2010
Accepted 9 November 2010
Available online 30 November 2010

Keywords:
Circulant graph
Well-covered graph
Independence polynomial
Powers of cycles
Cubic graphs

ABSTRACT

A graph is well-covered if every independent set can be extended to a maximum independent set. We show that it is co-*NP*-complete to determine whether an arbitrary graph is well-covered, even when restricted to the family of circulant graphs. Despite the intractability of characterizing the complete set of well-covered circulant graphs, we apply the theory of independence polynomials to show that several families of circulants are indeed well-covered. Since the lexicographic product of two well-covered circulants is also a well-covered circulant, our partial characterization theorems enable us to generate infinitely many families of well-covered circulants previously unknown in the literature.

1. Introduction

Let G be a graph. A subset T of the vertex set of G is an *independent set* if no vertices of T are adjacent in G. The *independence number* $\alpha(G)$ is the largest order of an independent set in G. We say that G is *well-covered* if every independent set is a subset of some (maximum) independent set with $\alpha(G)$ vertices. In other words, a graph is well-covered iff every maximal independent set is also a maximum independent set.

Given an arbitrary graph G, the problem of determining $\alpha(G)$ is *NP*-hard [11]. But in a well-covered graph, every independent set can be extended to a maximum independent set, and so $\alpha(G)$ can be trivially computed using the greedy algorithm. Hence, there is a polynomial-time algorithm to compute $\alpha(G)$ for any well-covered graph. Well-covered graphs were first introduced by Plummer in [18], who provides a comprehensive survey [19] of well-covered graphs and their properties.

One highly structured (and well-known) family of graphs are *circulants*. Given $n \geq 1$ and a *generating set* $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the circulant graph $C_{n,S}$ is the graph with vertex set $V = \mathbb{Z}_n$ such that for $u, w \in V$, uw is an edge of $C_{n,S}$ if and only if $|u - w|_n \in S$, where $|x|_n = \min\{|x|, n - |x|\}$ is the *circular distance* modulo n. For example, $C_{n,\{1\}}$ is the cycle C_n, while $C_{n,\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}}$ is the complete graph K_n. The circulant $C_{9,\{1,2\}}$ is illustrated in Fig. 1.

Circulant graphs are regular and vertex-transitive, and are a subset of the more general family of Cayley graphs. Specifically, circulant graphs are Cayley graphs over the simplest family of groups, namely the cyclic groups. Circulants arise in a variety of graph applications including the modeling of data connection networks [1,13] and the theory of designs and error-correcting codes [20]. In this paper, we investigate well-covered circulant graphs.

To give a simple illustration, we remark that C_6 is not well-covered because the maximal independent sets are $\{0, 3\}$, $\{1, 4\}$, $\{2, 5\}$, $\{0, 2, 4\}$, and $\{1, 3, 5\}$. On the other hand, C_7 is well-covered because each maximal independent set has order 3. For simple families of circulants, it is a relatively easy procedure to show that a graph is not well-covered — one just needs to find a maximal independent set with fewer than $\alpha(G)$ vertices.

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doi:10.1016/j.disc.2010.11.007

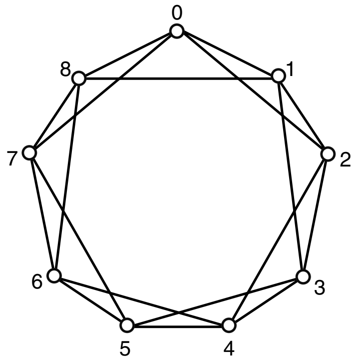

Fig. 1. The circulant graph $C_{9,\{1,2\}}$.

Table 1
Connected well-covered circulants on at most 12 vertices.

n	Generating sets S
4	{1}
5	{1}
6	{1, 3}, {2, 3}
7	{1}
8	{1, 3}, {1, 4 }
9	{1, 3}, {1, 2, 4}
10	{1, 4}, {2, 5}, {1, 2, 5}, {1, 3, 5}
11	{1, 2}, {1, 3}, {1, 2, 4}
12	{1, 4}, {3, 4}, {1, 2, 6}, {1, 3, 5}, {1, 3, 6}, {2, 3, 4}, {2, 3, 6}, {1, 4, 6}, {3, 4, 6}, {1, 2, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 2, 3, 5, 6}

For small values of n, one can generate all maximum independent sets to demonstrate whether $G = C_{n,S}$ is well-covered. Table 1 lists all connected non-isomorphic well-covered circulants on at most 12 vertices. We omit listing the complete graphs K_n since they are trivially well-covered.

While this brute-force approach is feasible for small values of n, it is not at all pragmatic for large n, especially when S is unstructured. For example, it is not clear whether the circulant graph

$$G^* = C_{150,\{1,2,3,12,13,14,16,17,18,27,28,29,31,32,33,42,43,44,46,47,48,57,58,59,60,61,62,63,72,73,74,75\}}$$

is well-covered. One certainly would not want to enumerate all maximum independent sets to determine whether this graph is well-covered. We require more sophisticated techniques, and we will develop them in the following sections.

More specifically, this paper makes two important contributions:

(a) We prove that it is co-*NP*-complete to determine whether an arbitrary circulant graph $G = C_{n,S}$ is well-covered. Therefore, it is unlikely that there is a polynomial-time algorithm to determine whether an arbitrary circulant $C_{n,S}$ is well-covered.

(b) We consider three families of graphs (powers of cycles, complements of powers of cycles, 3-regular circulants) and characterize all well-covered graphs within these three families. This provides a set of "building blocks" for generating an infinite family of well-covered circulants using the fact [22] that the lexicographic product of two well-covered circulants is also a well-covered circulant. For example, our analysis will show that the aforementioned graph G^* is well-covered since it is the lexicographic product of the well-covered circulants $C_{15,\{1,2,3\}}$ and $C_{10,\{4,5\}}$.

Our paper proceeds as follows. In Section 2, we establish the computational intractability of determining whether $G = C_{n,S}$ is well-covered. In Section 3, we introduce *independence polynomials*, a powerful tool that enumerates all independence sets of all orders in the form of a generating function, allowing us to formally prove that several complex families of circulants are not well-covered. We provide formulas for the independence polynomials of certain families of circulants, based on the results of a previous paper [3]. In Section 4, we apply the theory of independence polynomials to characterize several families of well-covered circulants and show how infinitely many families of well-covered circulants can be generated by applying the lexicographic product.

2. Computational intractability

It is co-*NP*-complete to decide whether an arbitrary graph G is well-covered [7,21]. When restricting G to the family of circulants, we might conjecture that the decision problem is computationally efficient, given the symmetric structure of circulant graphs. However, we show in this section that it is still co-*NP*-complete to decide whether an arbitrary circulant graph is well-covered. For discussion of relevant computational complexity, we refer the reader to [11]. To establish our result, we first require two lemmas and a definition.

Lemma 2.1 ([8]). *For all $n \in \mathbb{N}$, there are non-negative numbers $a_1, a_2, \ldots, a_n$, distinct mod $8^{\lceil \log_2 n \rceil} < 8n^3$, such that all sums $a_i + a_j$ are distinct mod $8^{\lceil \log_2 n \rceil} - 1$, and all sums $a_i + a_j + a_k$ are distinct mod $8^{\lceil \log_2 n \rceil} - 1$. Moreover, the sequence $a_1, a_2, \ldots, a_n$ is computable in time polynomial in n, and the distinctness claims remain true modulo any integer m satisfying $m > 3 \cdot (8^{\lceil \log_2 n \rceil} - 2)$.*

Definition 2.2. Let G be an arbitrary graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $C_{G,A}$ is a circulant on $N = 8^{\lceil \log_2 n \rceil} - 1$ vertices with generating set

$$S = \{|a_i - a_j|_N : v_i v_j \in E(G)\},$$

where $A = (a_1, a_2, \ldots, a_n)$ is an n-tuple $(a_1, a_2, \ldots, a_n)$ of positive integers satisfying the conditions of Lemma 2.1.

As an example, let G be the graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_7\}$ and edge set $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_1v_6, v_1v_7, v_3v_7, v_4v_7\}$. A 7-tuple satisfying the conditions of Lemma 2.1 is

$$A = (a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (54, 113, 14, 27, 85, 92, 36),$$

from which we derive a circulant $C_{G,A}$:

$$C_{G,A} = C_{511,\{7,9,13,18,22,38,58,59,99\}}.$$

Note that by Lemma 2.1, there is a polynomial-time algorithm to determine an n-tuple $(a_1, a_2, \ldots, a_n)$ satisfying the conditions of this lemma. Also note that there may be more than one $C_{G,A}$ that satisfies the conditions, but any such $C_{G,A}$ will do.

We remark that any vertex w of $C_{G,A}$ adjacent to $v = 0$ satisfies $w = a_i - a_j \pmod N$ for some (i, j) with $v_i v_j \in E(G)$. In this context, we say that the edge $v_i v_j$ of G *corresponds* to the vertex $w = a_i - a_j \pmod N$ in $C_{G,A}$. From now on, we will assume that A is an arbitrarily chosen n-tuple satisfying the conditions of Lemma 2.1, and so we will abbreviate $C_{G,A}$ by C_G.

Lemma 2.3 ([8]). *Let $w_1, w_2, \ldots, w_k$ be a k-clique in C_G, with $w_1 = 0$. Then for all $2 \le i \le k$, the edge e_i of G corresponding to w_i in C_G is adjacent to a certain vertex of G independent of i. Moreover, if $w_i = a_p - a_q \pmod N$ and $w_j = a_r - a_s \pmod N$, then $p = r$ or $q = s$.*

As shown in [8], this lemma follows quickly from Lemma 2.1. Let us use our earlier example to illustrate Lemma 2.3. By inspection, $\{0, 13, 22\}$ is a 3-clique in $C_G = C_{511,\{7,9,13,18,22,38,58,59,99\}}$. Notice that $w_2 = 13 = a_4 - a_3$ and $w_3 = 22 = a_7 - a_3$, i.e., e_2 and e_3 share the common vertex v_3 in G.

Since w_2 and w_3 are also adjacent in C_G, it follows that $\{v_3, v_4, v_7\}$ must be a 3-clique in G. In general, if C_G has a k-clique, then this produces a k-clique in G [8]. In the following lemma, we prove that a maximal k-clique of G corresponds to a maximal k-clique of C_G, and vice-versa.

Lemma 2.4. *There exists a maximal k-clique in G iff there exists a maximal k-clique in C_G.*

Proof. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be a maximal k-clique in C_G. By the vertex-transitivity of C_G, we can assume that $w_1 = 0$ without loss of generality. By Lemma 2.3, $w_j = a_m - a_{i_j} \pmod N$ for each $2 \le j \le k$ and some fixed index m. This implies that $T = \{v_m, v_{i_2}, v_{i_3}, \ldots, v_{i_k}\}$ is a k-clique in G. Now suppose that this k-clique is not maximal. Then we can add a new vertex v_q that is adjacent to each vertex in T, producing a $(k+1)$-clique in G. Let $w_{k+1} = a_m - a_q \pmod N$. Clearly, w_{k+1} is distinct from the previous k vertices of W. Then $\{w_1, w_2, \ldots, w_k, w_{k+1}\}$ is a $(k+1)$-clique in C_G, contradicting the maximality assumption.

Now we prove the converse. Let $\{v_1, v_2, \ldots, v_k\}$ be a maximal k-clique in G. Let $w_j = a_j - a_1 \pmod N$ for each $1 \le j \le k$. Then $S = \{w_1, w_2, w_3, \ldots, w_k\}$ is a k-clique in C_G, with $w_1 = 0$. Suppose that this k-clique is not maximal. Then we can add a new vertex w_{k+1} that is adjacent to each vertex in S, producing a $(k+1)$-clique in C_G. By Lemma 2.3, w_{k+1} must have the vertex label $a_q - a_1 \pmod N$, for some $v_q \in V(G)$, distinct from all of the other v_i's. Then $\{v_1, v_2, \ldots, v_k, v_q\}$ is a $(k+1)$-clique in G, contradicting the maximality assumption.

Therefore, we have proven that there exists a maximal k-clique in G iff there exists a maximal k-clique in C_G. □

Theorem 2.5. *Let $G = C_{n,S}$ be an arbitrary circulant graph. Then it is co-NP-complete to determine whether G is well-covered.*

Proof. Say that a graph belongs to the family $\mathcal{F}'$ if it is isomorphic to some $C_{G,A}$, where G is a graph on n vertices and $A = (a_1, a_2, \ldots, a_n)$ is an n-tuple satisfying the conditions of Lemma 2.1. Now let $\mathcal{F}$ be the family of all circulant graphs. Since each $C_{G,A}$ is a circulant, it follows that $\mathcal{F}' \subset \mathcal{F}$.

It is *NP*-complete to decide if an arbitrary graph G is not well-covered [7,21]. Thus, it is *NP*-complete to determine the existence of a maximal k-clique and maximal l-clique in an arbitrary graph G, for some $k \ne l$. By Lemma 2.4, it is *NP*-complete to determine the existence of a maximal k-clique and maximal l-clique in the corresponding graph C_G, for some $k \ne l$.

Therefore, if we restrict our circulants to just the family $\mathcal{F}'$, it is *NP*-complete to determine if an arbitrary graph in this family is not well-covered. This implies that the decision problem is co-*NP*-complete. Since $\mathcal{F}'$ is a subset of the set of all circulants, we conclude that it is co-*NP*-complete to determine whether an arbitrary circulant graph is well-covered. □

3. Independence polynomials

The *independence polynomial* of a graph G on n vertices is

$$I(G,x)=\sum_{k=0}^{n} i_k x^k,$$

where i_k is the number of independent sets of order k in G. By definition, the degree of $I(G,x)$ is just the independence number $\alpha(G)$. For example,

$$I(C_6,x)=1+6x+9x^2+2x^3,$$

as C_6 has $i_0=1$ (the empty set), $i_1=6$, $i_2=9$ (the number of non-edges of G), and $i_3=2$. The latter follows as there are precisely two independent sets of order 3, namely $\{0,2,4\}$ and $\{1,3,5\}$.

The independence polynomial of well-covered graphs has been a topic of interest [2,14,16,17], especially as it relates to its unimodality (in a *unimodal* polynomial, the coefficients of $I(G,x)$ are increasing up to a certain term, then decreasing after that term). Motivated by the literature connecting independence polynomials to well-covered graphs, we investigate well-covered circulant graphs, and apply the theory of independence polynomials to determine necessary and sufficient conditions for certain families of circulants to be well-covered.

The strategy is outlined as follows: To prove that $G=C_{n,S}$ is not well-covered, we first determine the coefficient of $x^{\alpha(G)}$ in $I(G,x)$, denoted by $[x^{\alpha(G)}]I(G,x)$. This proves that there are $[x^{\alpha(G)}]I(G,x)$ maximum independent sets. We then employ any enumeration technique (one method, based on *difference sequences*, is described at the end of this section) to generate these maximum independent sets. And once we have generated $[x^{\alpha(G)}]I(G,x)$ maximum independent sets, we can immediately stop because we know that there cannot be any more. Finally, we prove the existence of a smaller independent set I' that is not a subset of any of these maximum independent sets, proving that $G=C_{n,S}$ is not well-covered.

We now present the formulas for $I(G,x)$ when G is a power of a cycle or the complement of a power of a cycle. The dth power of the cycle C_n is the circulant graph $C_{n,\{1,2,\ldots,d\}}$. Powers of cycles have been a rich area of study with important applications to the analysis of perfect graphs (cf. [5,6,15]).

Theorem 3.1 ([3]). *Let n and d be integers with $n\ge 2d$ and $d\ge 1$. Then,*

$$I(C_{n,\{1,2,\ldots,d\}},x)=\sum_{k=0}^{\lfloor\frac{n}{d+1}\rfloor}\frac{n}{n-dk}\binom{n-dk}{k}x^k.$$

Theorem 3.2 ([3]). *Let n and d be integers with $n\ge 2d+2$ and $d\ge 1$. Let $r=n-2d-2$. Then,*

$$I(C_{n,\{d+1,d+2,\ldots,\lfloor\frac{n}{2}\rfloor\}},x)=1+\sum_{l=0}^{\lfloor\frac{d}{r+2}\rfloor}\frac{n}{2l+1}\binom{d-lr}{2l}x^{2l+1}(1+x)^{d-l(r+2)}.$$

As a simple corollary of Theorem 3.2, we have a formula for the number of maximum independent sets:

Corollary 3.3. *Let n and d be integers with $n\ge 2d+2$ and $d\ge 1$. Define $B_{n,d}=C_{n,\{d+1,d+2,\ldots,\lfloor\frac{n}{2}\rfloor\}}$. Then, $\alpha(B_{n,d})=d+1$. Furthermore, $[x^{d+1}]I(B_{n,d},x)=2^{\frac{n}{2}}$ if $n=2d+2$ and $[x^{d+1}]I(B_{n,d},x)=n$ if $n>2d+2$.*

Proof. If $n=2d+2$, then $B_{n,d}=C_{2d+2,\{d+1\}}$, and so $B_{n,d}$ is just $d+1$ disjoint copies of the trivial graph K_2. Hence, $I(B_{n,d},x)=(1+2x)^{d+1}$, implying that $[x^{d+1}]I(B_{n,d},x)=2^{d+1}=2^{\frac{n}{2}}$. Thus, assume that $n>2d+2$, i.e., $r=n-2d-2>0$.

For each $0\le l\le\lfloor\frac{d}{r+2}\rfloor$, our formula for $I(B_{n,d},x)$ adds a polynomial of degree $2l+1+d-l(r+2)=d-lr+1$. Thus, $\alpha(B_{n,d})=\deg(I(B_{n,d},x))=d+1$. Furthermore, x^{d+1} terms appear in our polynomial precisely when $l=0$ or $r=0$. From our assumption that $r>0$, an x^{d+1} term can only appear when $l=0$. From this, we immediately derive the desired result that $[x^{d+1}]I(B_{n,d},x)=\frac{n}{0+1}\binom{d-0}{0}=n$. □

For our analysis of cubic (i.e., 3-regular) graphs, we require the following two results:

Theorem 3.4 ([9]). *Let $G=C_{2n,\{a,n\}}$ for some $1\le a<n$. Let $t=\gcd(2n,a)$.*

(a) *If $\frac{2n}{t}$ is even, then G is isomorphic to t copies of $C_{\frac{2n}{t},\{1,\frac{n}{t}\}}$.*

(b) *If $\frac{2n}{t}$ is odd, then G is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t},\{2,\frac{2n}{t}\}}$.*

Theorem 3.5 ([12]). *Let $G=C_{2n,\{a,n\}}$ for some $1\le a<n$. Let $t=\gcd(2n,a)$.*

(a) *If* $\frac{n}{t}$ *is even, then* $I(G, x) = (I(C_{\frac{2n}{t},\{\frac{n}{t}-1,\frac{n}{t}\}}, x))^t$.

(b) *If* $\frac{2n}{t}$ *is even and* $\frac{n}{t}$ *is odd, then* $I(G, x) = (I(C_{\frac{2n}{t},\{\frac{n}{t}-1,\frac{n}{t}\}}, x) + 2x^{\frac{n}{t}})^t$.

(c) *If* $\frac{2n}{t}$ *is odd, then* $I(G, x) = (I(C_{\frac{4n}{t},\{\frac{2n}{t}-1,\frac{2n}{t}\}}, x))^{\frac{t}{2}}$.

We conclude this section by defining the intuitive concept of *difference sequences*.

Definition 3.6. Let $G = C_{n,S}$ be a circulant graph. For each k-set $\{v_1, v_2, \ldots, v_k\}$ of the vertices of G, with $0 \le v_1 < v_2 < \cdots < v_k \le n - 1$, the difference sequence is

$$D = (d_1, d_2, \ldots, d_k) = (v_2 - v_1, v_3 - v_2, \ldots, v_k - v_{k-1}, n + v_1 - v_k).$$

We can visualize difference sequences as follows: Spread the n vertices clockwise around a circle, and highlight the k chosen vertices $v_1, v_2, \ldots, v_k$. For each $1 \le i \le k$, let d_i be the number of clockwise steps required to move from v_i to v_{i+1}, where $v_{k+1} := v_1$. By this reasoning, it is clear that $\sum_{i=1}^{k} d_i = n$ and that $v_j = v_1 + \sum_{i=1}^{j-1} d_i$ for each $1 \le j \le k$. We now provide two more definitions: *cyclic subsequences* and *valid* difference sequences.

Definition 3.7. Let i and j be integers with $1 \le i, j \le k$. Then the cyclic subsequence of the difference sequence D from term i to term j is

$$D_{i,j} := \begin{cases} (d_i, d_{i+1}, \ldots, d_{j-1}, d_j) & \text{if } i \le j, \\ (d_i, d_{i+1}, \ldots, d_k, d_1, d_2, \ldots, d_j) & \text{if } i > j. \end{cases}$$

Definition 3.8. A difference sequence $D = (d_1, d_2, \ldots, d_k)$ of $C_{n,S}$ is valid if there does not exist a cyclic subsequence $D_{i,j}$ whose terms sum to an element in S.

From these definitions, the following result is clear.

Theorem 3.9. *Let I be a set of k vertices in $C_{n,S}$, and let D_I be the corresponding difference sequence. Then I is independent in $C_{n,S}$ iff D_I is valid in $C_{n,S}$.*

By Theorem 3.9, there is a simple correspondence between independent sets and valid difference sequences. For notational convenience, we will write D_I as D, since I will be clear in all situations. We are now ready to present our characterization theorems.

4. Characterization theorems for three families of circulants

In this section, we develop characterization theorems for three families of circulants: powers of cycles, complements of powers of cycles, and cubic graphs. We first begin by classifying all well-covered circulants that are powers of cycles.

It is well-known (cf. [10]) that C_n is well-covered iff $n \le 5$ or $n = 7$. The following theorem generalizes this result.

Theorem 4.1. *Let n and d be integers with $n \ge 2d$ and $d \ge 1$. Then, $C_{n,\{1,2,\ldots,d\}}$ is well-covered iff $n \le 3d + 2$ or $n = 4d + 3$.*

Proof. Let $G = C_{n,\{1,2,\ldots,d\}}$. By Theorem 3.1, $\alpha(G) = \lfloor \frac{n}{d+1} \rfloor = p$, for some integer p. Then, $n = (d + 1)p + q$, for some $0 \le q \le d$. If $p = 1$, then $n = 2d$ or $n = 2d + 1$, implying that $G = K_n$. Thus, G is trivially well-covered in this case. If $p = 2$, then $2d + 2 \le n \le 3d + 2$. Since $\{i, i + d + 1\}$ is a maximum independent set for each i (where addition is taken mod n), each vertex appears in some maximum independent set. So G is well-covered in this case as well.

It remains to deal with the case $p \ge 3$. Let a and b be the unique pair of integers such that $n = a(p - 1) + b$, where $0 \le b \le p - 2$. Since $n = (d + 1)p + q = a(p - 1) + b$, we have $p(a - d) = q + a + (p - b) > 0$, implying that $a \ge d + 1$. Consider the difference sequence

$$D' = (\underbrace{a, a, \ldots, a}_{p-b-1}, \underbrace{a + 1, a + 1, \ldots, a + 1}_{b}).$$

The sum of the elements of D' is $(p - b - 1)a + b(a + 1) = n$. Each term in the sequence is at least $d + 1$. Thus, D' is a valid difference sequence of G with $p - 1$ elements. Let $I' = \{v_1, v_2, \ldots, v_{p-1}\}$ be the (unique) set of vertices with difference sequence D' and $v_1 = 0$. By Theorem 3.9, I' is an independent set of order $p - 1$.

If G is well-covered, then there exists an independent set I of order $\alpha(G) = p$ with $I' \subset I$. Let $I = I' \cup \{w\}$, for some $0 \le w \le n - 1$. Then w lies between two consecutive vertices v_j and v_{j+1}, for some $1 \le j \le p - 1$. (If $j = p - 1$, we let $v_p = v_1 + n = n$). Then $v_{j+1} - v_j = d_j \in \{a, a + 1\}$, implying that $v_{j+1} - v_j \le a + 1$. Since $\{v_j, w, v_{j+1}\}$ is an independent set in $G = C_{n,\{1,2,\ldots,d\}}$, $v_{j+1} - v_j = (v_{j+1} - w) + (w - v_j) \ge |v_{j+1} - w|_n + |w - v_j|_n \ge (d + 1) + (d + 1)$. This implies that $a+1 \geq (d+1)+(d+1)$. In other words, a necessary condition for G to be well-covered is $a+1 \geq 2d+2$, which is equivalent to

$$a = \left\lfloor \frac{(d+1)p+q}{p-1} \right\rfloor \geq 2d+1.$$

For G to be well-covered, we must have $a \geq 2d+1$. First suppose that $p \geq 4$. Then,

$$a = \left\lfloor \frac{(d+1)p+q}{p-1} \right\rfloor \leq \frac{(d+1)p+d}{p-1} = d+1+\frac{2d+1}{p-1} \leq d+1+\frac{2d+1}{3} \leq 2d+1,$$

with equality iff $(p, d, q) = (4, 1, 1)$. This case (which corresponds to $G = C_9$) is not well-covered; this is easily seen by noting that $\{0, 3, 6\}$ is a maximal independent set that is not maximum. In all other cases, we have established a contradiction. Thus, G is not well-covered if n and d satisfy $\alpha(G) = p = \lfloor \frac{n}{d+1} \rfloor \geq 4$.

Now suppose $p = 3$. Then if $q \leq d-2$, then

$$a = \left\lfloor \frac{(d+1)p+q}{p-1} \right\rfloor = \left\lfloor \frac{3(d+1)+q}{2} \right\rfloor \leq \left\lfloor \frac{3(d+1)+(d-2)}{2} \right\rfloor = \left\lfloor \frac{4d+1}{2} \right\rfloor < 2d+1.$$

Hence, if $p = 3$ and G is well-covered, then we must have $q = d$ or $q = d-1$. Thus, the only possible well-covered graphs occur in the cases $(p, q) = (3, d)$ and $(p, q) = (3, d-1)$. These pairs correspond to the circulants $G = C_{4d+3,\{1,2,\ldots,d\}}$ and $G = C_{4d+2,\{1,2,\ldots,d\}}$, respectively. We prove that the former is well-covered, while the latter is not.

Consider the graph $G = C_{4d+3,\{1,2,\ldots,d\}}$. By Theorem 3.1, $\alpha(G) = \lfloor \frac{4d+3}{d+1} \rfloor = 3$. We show that every maximal independent set is of order 3. Let I' be an independent 2-set. Without loss, let $I' = \{0, x\}$, for some $x \in [d+1, 3d+2]$. If $d+1 \leq x \leq 2d+1$, then $I = \{0, x, x+d+1\}$ is an independent 3-set of G. If $2d+2 \leq x \leq 3d+2$, then $I = \{0, d+1, x\}$ is an independent 3-set of G. In both cases, I' can be extended to a maximum independent set I. Thus, we have shown that every maximal independent set is of order $p = 3$, proving that G is well-covered.

Now consider the graph $G = C_{4d+2,\{1,2,\ldots,d\}}$. By Theorem 3.1, $\alpha(G) = \lfloor \frac{4d+2}{d+1} \rfloor = 3$. Thus, every maximal independent set must have three vertices. But $I' = \{0, 2d+1\}$ is a maximal independent set that is not maximum. Hence, G is not well-covered in this case.

We conclude that if $p \geq 3$, then G is well-covered only for the case $(p, q) = (3, d)$, i.e., when $n = 4d+3$. If $p \leq 2$, then $n \leq 3d+2$, and G is trivially well-covered in each of these cases. We conclude that $G = C_{n,\{1,2,\ldots,d\}}$ is well-covered iff $n \leq 3d+2$ or $n = 4d+3$. □

We have now given a full characterization of well-covered powers of cycles. We now determine which complements of powers of cycles are well-covered.

Theorem 4.2. *Let n and d be integers with $n \geq 2d+2$ and $d \geq 1$. Then $C_{n,\{d+1,d+2,\ldots,\lfloor \frac{n}{2} \rfloor\}}$ is well-covered iff $n > 3d$ or $n = 2d+2$.*

Proof. First note that $G = C_{n,\{d+1,d+2,\ldots,\lfloor \frac{n}{2} \rfloor\}}$ is clearly well-covered if $n = 2d+2$, since G is simply $d+1$ isomorphic copies of K_2. Thus, we can assume that $n > 2d+2$.

By Corollary 3.3, $\alpha(G) = d+1$. Clearly, the difference sequence $(\underbrace{1, 1, \ldots, 1}_{d}, n-d)$ is valid. By Theorem 3.9, this gives rise to n independent sets of order $d+1$, namely the sets $S_i = \{i, i+1, i+2, \ldots, i+d\}$, for each $i = 0, \ldots, n-1$, where the elements are reduced mod n. By Corollary 3.3, $[x^{d+1}]I(G, x) = n$, and so there cannot be any other maximum independent sets. Therefore, if G is well-covered, then every independent set must be a subset of some S_i for some $0 \leq i \leq n-1$.

First, consider the case $n > 3d$. Let I' be any independent set $\{v_1, v_2, \ldots, v_k\}$, with $k < d+1$. Without loss, assume that $v_1 = 0$ and $0 < v_2 < v_3 < \cdots < v_k \leq n-1$. Since I' is independent in G, no $v_i \in [d+1, n-d-1]$. So each $v_i \leq d$ or $v_i \geq n-d$. If $v_2 \geq n-d$, then $I' \subset S_{n-d}$. If $v_k \leq d$, then $I' \subset S_0$. In both cases, I' is contained in a maximum independent set. So the only other case to consider is when $v_2 \leq d$ and $v_k \geq n-d$. In this situation, there is a unique index j for which $v_j \leq d$ and $v_{j+1} \geq n-d$. Since $v_{j+1} - v_j \geq n-2d > d$, we must have $|v_j - v_{j+1}|_n = n + v_j - v_{j+1} \leq d$ for I' to be an independent set. This implies that $v_{j+1} \geq n + v_j - d$. Thus, $I' \subset S_{n+v_j-d} = \{0, 1, \ldots, v_j, n+v_j-d, \ldots, n-1\}$. This establishes that G is well-covered, for any (n, d) with $n > 3d$.

Finally, consider the case $2d+3 \leq n \leq 3d$. In this case, $d \geq 3$. The set $I' = \{0, d, n-d\}$ is independent in G, since the circular distances are d, $n-d$, and $n-2d \leq d$, none of which appears in the generating set $S = \{d+1, d+2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. But I' cannot be contained in a maximum independent set (note $d+1 > 3$), since there is no i for which $I' \subset S_i$. Hence, I' cannot be extended to a maximum independent set, and so G is not well-covered. Thus, we have proved that $G = C_{n,\{d+1,d+2,\ldots,\lfloor \frac{n}{2} \rfloor\}}$ is well-covered iff $n > 3d$ or $n = 2d+2$. □

We now determine the set of well-covered cubic circulants, i.e., circulant graphs of degree 3. Our proof uses independence polynomials. We remark that an alternative proof appears in [12] using a characterization theorem of all well-covered cubic graphs [4]. We first limit our analysis to connected cubic circulants (e.g. $C_{6,\{1,3\}}$) and then expand the analysis to all cubic circulants (e.g. $C_{12,\{2,6\}}$).

Theorem 4.3. *Let G be a connected circulant cubic graph. Then G is well-covered iff it is isomorphic to one of the following graphs: $C_{4,\{1,2\}}$, $C_{6,\{1,3\}}$, $C_{6,\{2,3\}}$, $C_{8,\{1,4\}}$, or $C_{10,\{2,5\}}$.*

First, we remark that each of the above five circulant graphs is well-covered. This is seen from Table 1. We now prove that no other connected 3-regular circulant is well-covered.

Every connected 3-regular circulant G is isomorphic to $C_{2m,\{a,m\}}$, for some $1 \le a < m$. It is straightforward to see that G is not connected iff $\gcd(a, m) > 1$. So we must have $\gcd(a, m) = 1$. By Theorem 3.4, every connected 3-regular circulant must be isomorphic to one of the following graphs: $C_{4n,\{1,2n\}}$, $C_{4n+2,\{1,2n+1\}}$, or $C_{4n+2,\{2,2n+1\}}$. Let us consider each of these cases separately.

Case 1. $G = C_{4n,\{1,2n\}}$.

G is well-covered for $n \le 2$ so suppose $n \ge 3$. By Theorem 3.5,

$$I(G, x) = I(C_{4n,\{1,2n\}}, x) = I(C_{4n,\{2n-1,2n\}}, x).$$

By Corollary 3.3, $\alpha(G) = \deg(I(G, x)) = \deg(I(C_{4n,\{2n-1,2n\}}, x)) = 2n - 1$. To complete the proof, it suffices to find one maximal independent set I whose cardinality is less than $2n - 1$.

If $n = 3$, let $I = \{0, 4, 8\}$. If $n = 3k + 1$ for $k \ge 1$, let $I = \{0, 3, 6, \dots, 12k\}$. If $n = 3k + 2$ for $k \ge 1$, let $I = \{0, 3, 6, \dots, 12k+6\}$. Finally, if $n = 3k+3$ for $k \ge 1$, then let $I = \{0, 3, 6, \dots, 6k+3, 6k+5, 6k+8, 6k+11, \dots, 12k+8, 12k+10\}$. In all cases, I is a maximal independent set with $|I| < 2n - 1$.

Therefore, $G = C_{4n,\{1,2n\}}$ is well-covered iff $n = 1$ or $n = 2$.

Case 2. $G = C_{4n+2,\{1,2n+1\}}$.

G is well-covered for $n = 1$ so suppose $n \ge 2$. By Theorem 3.5,

$$I(G, x) = I(C_{4n+2,\{1,2n+1\}}, x) = I(C_{4n+2,\{2n,2n+1\}}, x) + 2x^{2n+1}.$$

By Corollary 3.3, $\alpha(G) = \deg(I(G, x)) = \max(2n, 2n + 1) = 2n + 1$.

Let $D = (d_1, d_2, \dots, d_{2n+1})$ be a valid difference sequence. Clearly we require $d_i = 2$ for each i. Therefore, there are two possible maximum independent sets, the set of even vertices and the set of odd vertices. Now let $I' = \{0, 3\}$. Since I' cannot be extended to one of these maximum independent sets, we conclude that $G = C_{4n+2,\{1,2n+1\}}$ is well-covered iff $n = 1$.

Case 3. $G = C_{4n+2,\{2,2n+1\}}$.

G is well-covered for $n \le 2$ so suppose $n \ge 3$. By Theorem 3.5,

$$I(G, x) = I(C_{4n+2,\{2,2n+1\}}, x) = I(C_{4n+2,\{2n,2n+1\}}, x).$$

By Corollary 3.3, $\alpha(G) = \deg(I(G, x)) = \deg(I(C_{4n+2,\{2n,2n+1\}}, x)) = 2n$.

To complete the proof, it suffices to find one maximal independent set I whose cardinality is less than $2n$. If $n = 3$, let $I = \{0, 3, 6, 9\}$. If $n = 4$, let $I = \{0, 1, 5, 6, 11, 12\}$. If $n = 3k - 1$ for $k \ge 2$, let $I = \{0, 3, 6, \dots, 12k - 6, 12k - 5\}$. If $n = 3k$ for $k \ge 2$, let $I = \{0, 3, 6, \dots, 12k - 3, 12k - 2\}$. Finally, if $n = 3k + 1$ for $k \ge 2$, then let $I = \{0, 3, 6, \dots, 6k, 6k + 1, 6k + 4, 6k + 7, \dots, 12k + 1, 12k + 2, 12k + 5\}$. In all cases, I is a maximal independent set with $|I| < 2n$.

Therefore, we have shown that $G = C_{4n+2,\{2,2n+1\}}$ is well-covered iff $n = 1$ or $n = 2$.

We conclude that there are only five connected well-covered circulants of the form $G = C_{2n,\{a,n\}}$, where $\gcd(a, n) = 1$. These circulants are isomorphic to one of the following: $C_{4,\{1,2\}}$, $C_{6,\{1,3\}}$, $C_{6,\{2,3\}}$, $C_{8,\{1,4\}}$, $C_{10,\{2,5\}}$. This completes the proof. $\square$

As an immediate corollary, we have a characterization of all well-covered 3-regular circulants.

Theorem 4.4. *Let $G = C_{2n,\{a,n\}}$, where $1 \le a < n$. Let $t = \gcd(2n, a)$. Then G is well-covered iff $\frac{2n}{t} \in \{3, 4, 5, 6, 8\}$.*

Proof. First note that if G is the disjoint union of k components, then G is well-covered iff each of the k components are well-covered. We consider two cases: $\frac{2n}{t}$ even and $\frac{2n}{t}$ odd. By Theorem 3.4, if $\frac{2n}{t}$ is even, then $G = C_{2n,\{a,n\}}$ is isomorphic to t copies of $C_{\frac{2n}{t},\{1,\frac{n}{t}\}}$. By Theorem 4.3 and the comment above, G is well-covered iff $\frac{2n}{t}$ is 4, 6, or 8.

By Theorem 3.4, if $\frac{2n}{t}$ is odd, then $G = C_{2n,\{a,n\}}$ is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t},\{2,\frac{2n}{t}\}}$. By Theorem 4.3 and the previous comment, G is well-covered iff $\frac{4n}{t}$ is 6 or 10. This establishes the desired result. $\square$

Therefore, we have found necessary and sufficient conditions for a graph $G = C_{n,S}$ to be well-covered, for each of our three families.

For any two graphs G and H, the lexicographic product is a new graph $G[H]$ with vertex set $V(G) \times V(H)$ such that any two vertices (g, h) and (g', h') in $G[H]$ are adjacent iff $(g \sim g')$ or $(g = g'$ and $h \sim h')$. The following results on the lexicographic product enables us to generate even more families of well-covered circulants.

Theorem 4.5 ([22]). *Let G and H be nonempty graphs. Then $G[H]$ is well-covered iff G and H are both well-covered.*

Theorem 4.6 ([12]). *Let G and H be circulants with $G = C_{n,S_1}$ and $H = C_{m,S_2}$. Define*

$$S = \left(\bigcup_{t=0}^{\lfloor \frac{m-1}{2} \rfloor} tn + S_1 \right) \bigcup \left(\bigcup_{t=1}^{\lfloor \frac{m}{2} \rfloor} tn - S_1 \right) \bigcup nS_2,$$

where $tn \pm S_1 = \{tn \pm r : r \in S_1\}$ and $nS_2 = \{nq : q \in S_2\}$. Then $G[H]$ is a circulant, and is isomorphic to $C_{nm,S}$.

If G and H are any of the well-covered circulants in Theorems 4.1, 4.2 and 4.4, then the lexicographic product $G[H]$ is also a well-covered circulant by Theorems 4.5 and 4.6. For example, the circulants $G = C_{15,\{1,2,3\}}$ and $H = C_{10,\{4,5\}}$ are both well-covered, and so Theorem 4.6 implies that the circulant

$$G[H] = C_{150,\{1,2,3,12,13,14,16,17,18,27,28,29,31,32,33,42,43,44,46,47,48,57,58,59,60,61,62,63,72,73,74,75\}}$$

is also well-covered. By this technique, we can produce infinitely many more well-covered circulant graphs whose generating sets are highly unstructured.

Acknowledgements

The authors thank the referees for their insightful comments that significantly improved the presentation of this paper. This research was partially supported by a grant from NSERC.

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