Lecture 8

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Computer-Aided Reasoning, Lecture 8

SAT, NP-Completeness

- kSAT
  - Literals: variables or their negations
  - Clause: disjunction of literals
  - CNF formula (Conjunctive Normal Form): conjunction of clauses
  - kCNF: CNF formula w/ at most k literals per clause
    - kSAT: The set of satisfiable kCNF formulas
- Recall: SAT (= set of satisfiable propositional formulas) is NP-complete
  - NP: languages whose membership can be verified in P-time
  - NPC: Hardest problems in NP
  - A P-time algorithm for 1 NPC problem implies P-time algorithms for every problem in NP
- 3SAT is NP-complete (Cook 1971)
- SAT $\leq_p$ 3SAT: define P-time function F s.t. $x \in$ SAT iff $F(x) \in$ 3SAT

Tseitin Transformation

- Goal: define a P-time function F s.t. $x \in$ SAT iff $F(x) \in$ 3SAT
- Is there a P-time function F that given g, returns equivalent CNF formula?
  - No: just going from DNF to CNF is *provably* exponential in the worst case
- Idea: define a P-time (linear) function that preserves satisfiability
- Example: $(p \vee (q \wedge \neg r)) \wedge s$
- Introduce definition d: $d = q \wedge \neg r$
- Example is now in CNF: $(p \vee d) \wedge s$
- Turn definition into CNF: $(\neg d \vee q) \wedge (\neg d \vee \neg r) \wedge (\neg q \vee r \vee d)$
- Finally: $(p \vee d) \wedge s \wedge (\neg d \vee q) \wedge (\neg d \vee \neg r) \wedge (\neg q \vee r \vee d)$
- Not equivalent, but *equisatisfiable*
- Using this idea, we recursively introduce definitions as needed

2SAT is in P

- SAT and even 3SAT are NPC. What about 2SAT?
- Construct a graph corresponding to a 2CNF formula F
  - Vertices: $\{x, \neg x : x \text{ a variable}\}$
  - Edges: $\{(u, v),\ (\neg v, \neg u) : (\neg u \vee v) \in F\}$ (u, v are literals)
- Lemma 1: $u \rightarrow v$ (a path from u to v) implies $\neg v \rightarrow \neg u, u \Rightarrow v$
- Lemma 2: F is unsatisfiable iff there is a variable x, st:
  - there is a path from x to $\neg x$, and
  - there is a path from $\neg x$ to x
- Proof: Pong: (A ⇐ B) (lemma 1)
  - Ping: (¬A ⇐ ¬B) (which is equivalent to (A ⇒ B))
  - choose unassigned u s.t. no path to $\neg u$ exists; set u, reachable vertices to true; repeat; notice if $u \rightarrow x, \neg x$, then $\neg x \rightarrow \neg u$, so $u \rightarrow \neg u$
- 2SAT can be solved in linear time
- Run SCC: SAT iff no component contains a literal and its negation

2SAT Example

- $(p \vee q) \wedge (p \vee \neg q) \wedge \neg p$?
- Construct a graph corresponding to a 2CNF formula F
  - Vertices: $\{x, \neg x : x \text{ a variable}\}$
  - Edges: $\{(u, v),\ (\neg v, \neg u) : (\neg u \vee v) \in F\}$ (u, v are literals)
- Vertices: $\{p, \neg p, q, \neg q\}$
- Try it!
- Note: for unit clause $\neg p$, treat it as $\neg p \vee \neg p$
- Edges: $\{(\neg p, q),\ (\neg q, p),\ (\neg p, \neg q),\ (q, p),\ (p, \neg p)\}$
- SCC: $\{\{\neg p, q, p, \neg p\}\}$, so UNSAT

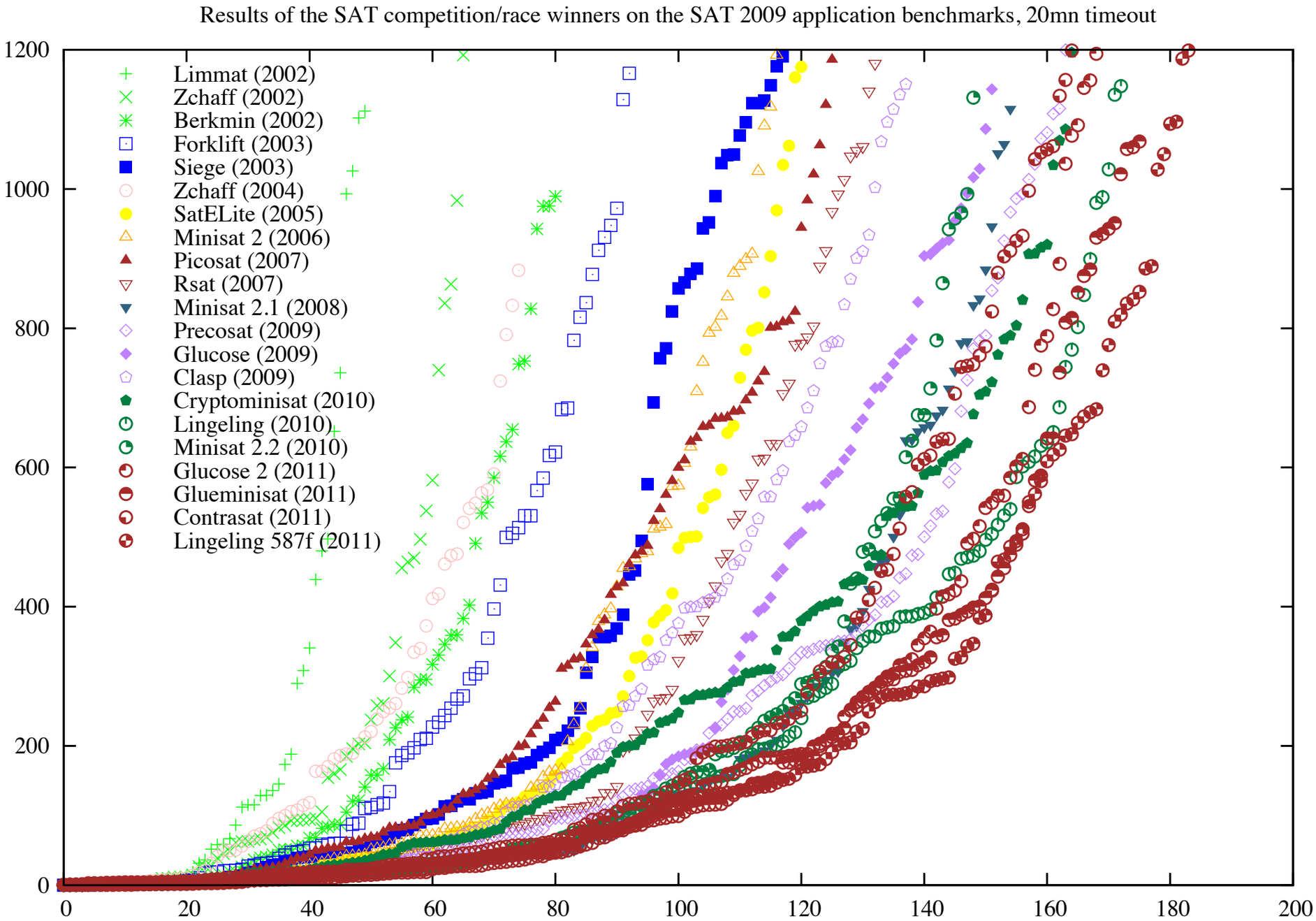
Short History

- Ancients: Logic invented as a scientific field of study by Aristotle (380-322 B.C.)
  - Categorical logic, quantifiers, 2-valued, *satisfiability, validity, …*
- Medieval Logicians: early ideas of mechanization, eg, Lull (1232-1315)
- Leibniz (1646-1716): calculus ratiocinator, a kind of calculating machine
- Stanhope (1753-1816): first machine to solve logic problems
- Boole (1815-1864): Boolean algebra
- Frege (1848-1925): Concept notation, basis for modern formal logic
- Russell & Whitehead, Godel, Herbrand, Pierce, Tarski, …
- Shannon (1940): Boolean logic to minimize circuits
- Davis & Putnam (1958): DP algorithm, DPLL (1962)BDDs (Lee 1959), …, ROBDDs (Bryant 1986, …)
- Bryant, Clarke, Emerson & McMillan received the 1998 Paris Kanellakis Award for “their invention of 'symbolic model checking', a method of formally checking system designs widely used in the computer hardware industry.”
- CDCL: decision heuristics, backjumping, learning/forgetting, restarts, pre/in-processing, …

SAT Algorithms

- SAT used everywhere: Logic, AI, SAT, scheduling, circuit design, game theory, verification (predicate abstraction), reliability theory, etc.
- Theory vs practice: NPC problem, so what do we do?
  - Option1: give up; not feasible
  - Option2: come up with techniques that work on interesting problems
- If the problem is important, we go with option2
- Multi-billion dollar industries are based on SAT
- An even more extreme example: termination analysis
- When reasoning about systems, SAT is one of the simplest problems we have to deal with

Cactus Plots



From: Le Berre&Biere 2011

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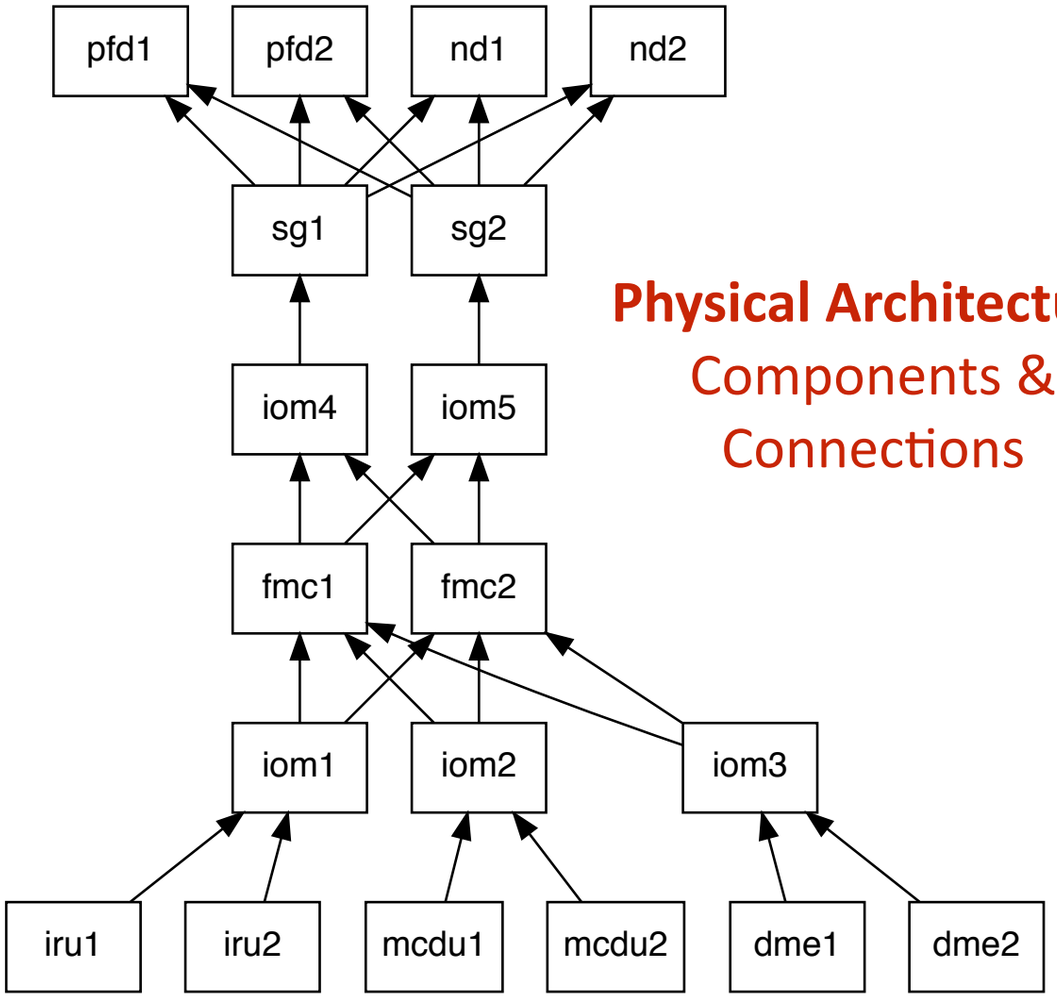
Safety Analysis Example

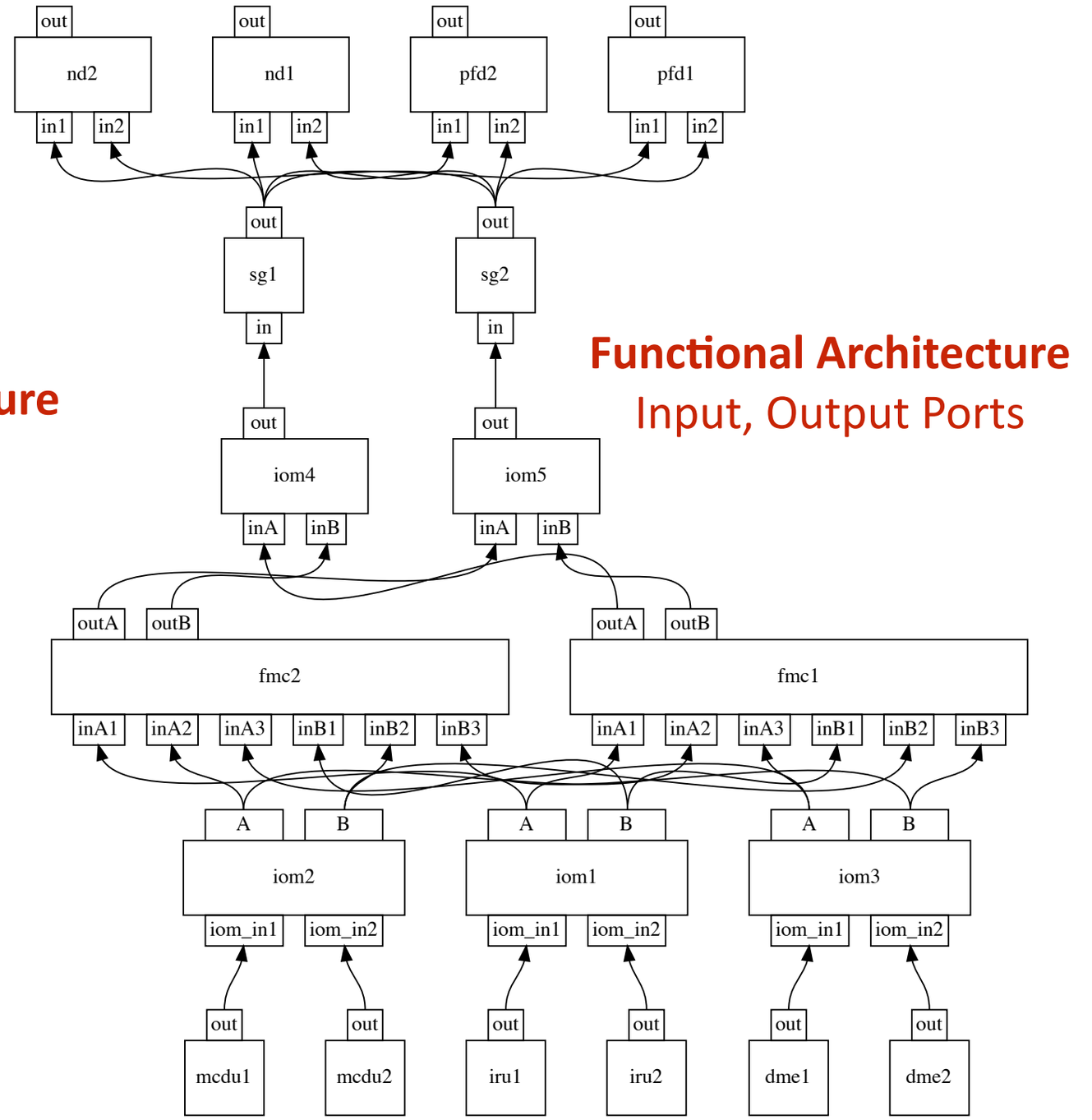
- The book has many interesting applications of SAT, e.g., circuits
- Let's look at one example in more depth which highlights some interesting issues: modeling, formula simplification, probability.
- See https://gitlab.com/pmanolios/safety-analysis
- Look at Webpage
- run make 777-example-pdf.out

777-Inspired Architecture

Modeling loss of display of navigation information



Physical Architecture
Components & Connections



Functional Architecture
Input, Output Ports

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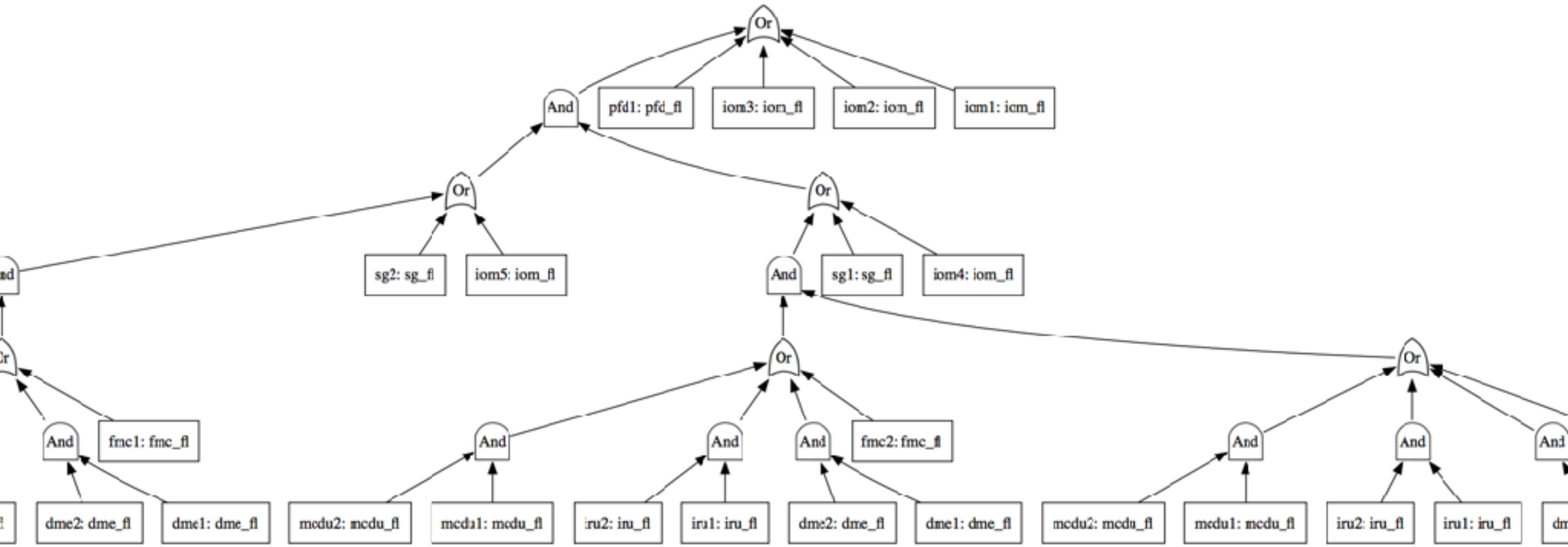
Automated Fault Tree Synthesis

Fault trees automatically synthesized from models
Model update, reuse, maintenance, verification, analysis capabilities



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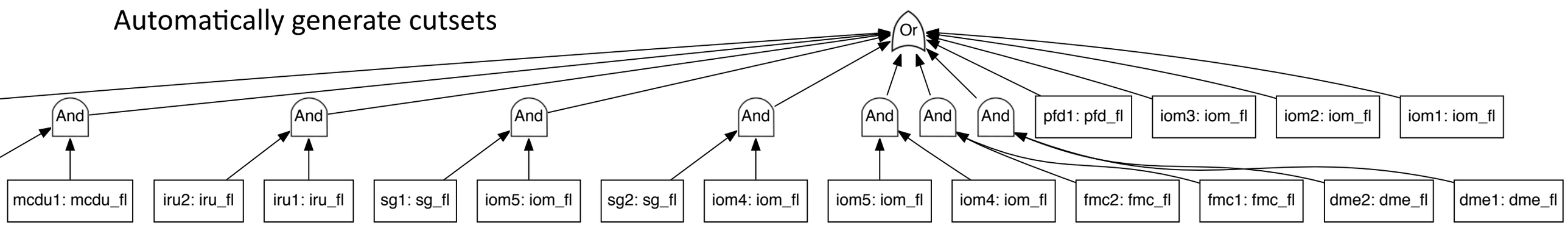
Architectural Analysis

Quantitative analysis:
Top level fault probability 3.0002e-06

Qualitative analysis:
Automatically generate cutsets



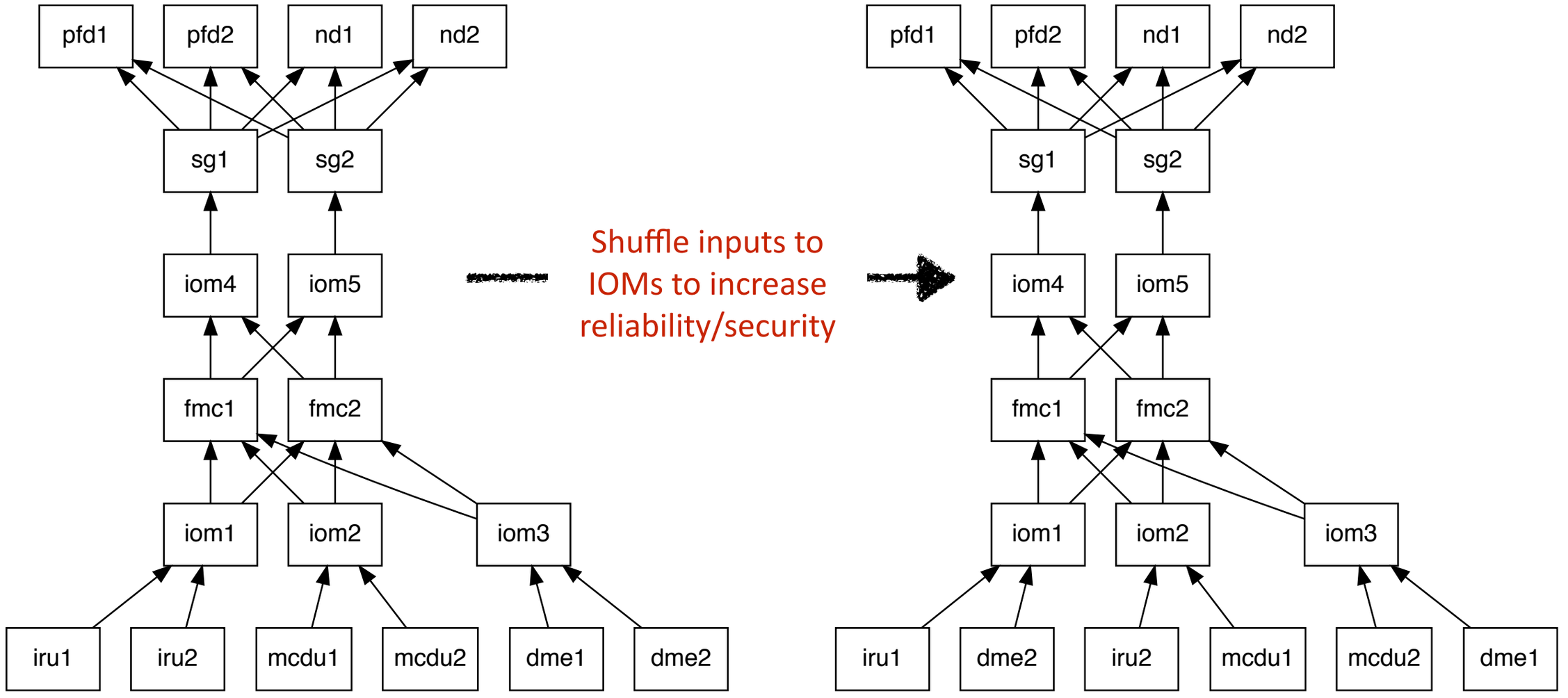
Automatically generate importance measures

Components	Probability	Importance Measure
iom1:iom_fl	9.9999E-07	0.3333
iom2:iom_fl	9.9999E-07	0.3333
iom3:iom_fl	9.9999E-07	0.3333
pfd1:pfd_fl	2.0000E-10	6.6662E-05
dme1:dme_fl; dme2:dme_fl	9.9999E-13	3.3331E-07

■ ■ ■

IOMs contribute most to top-level probability. **Remove the single points of failure.**

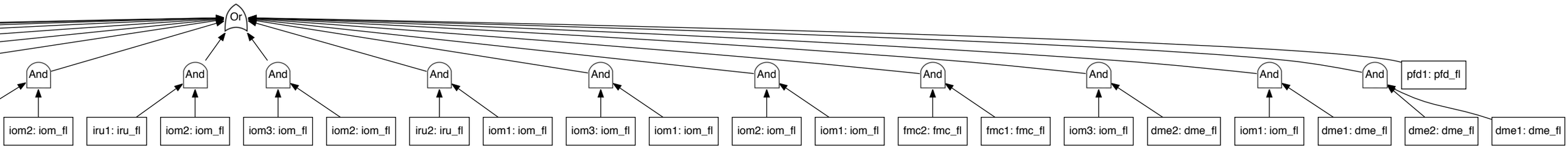
Architectural Trades/Synthesis



Architectural Analysis

Quantitative analysis:
Top level fault probability 2.1599e-10 (before: 3.0002e-06)

Qualitative analysis:
Automatically generate cutsets



Automatically generate importance measures

Components	Probability	Importance Measure
pfd1:pfd_fl	2.0000E-10	0.9259
dme1:dme_fl; dme2:dme_fl	9.9999E-13	0.0046
dme1:dme_fl; iom1:iom_fl	9.9999E-13	0.0046
iom1:iom_fl; iom3:iom_fl	9.9999E-13	0.0046
iru1:iru_fl; iru2:iru_fl	9.9999E-13	0.0046

IOMs are no longer single points of failure. **You need to compromise at least one other system for a failure.**

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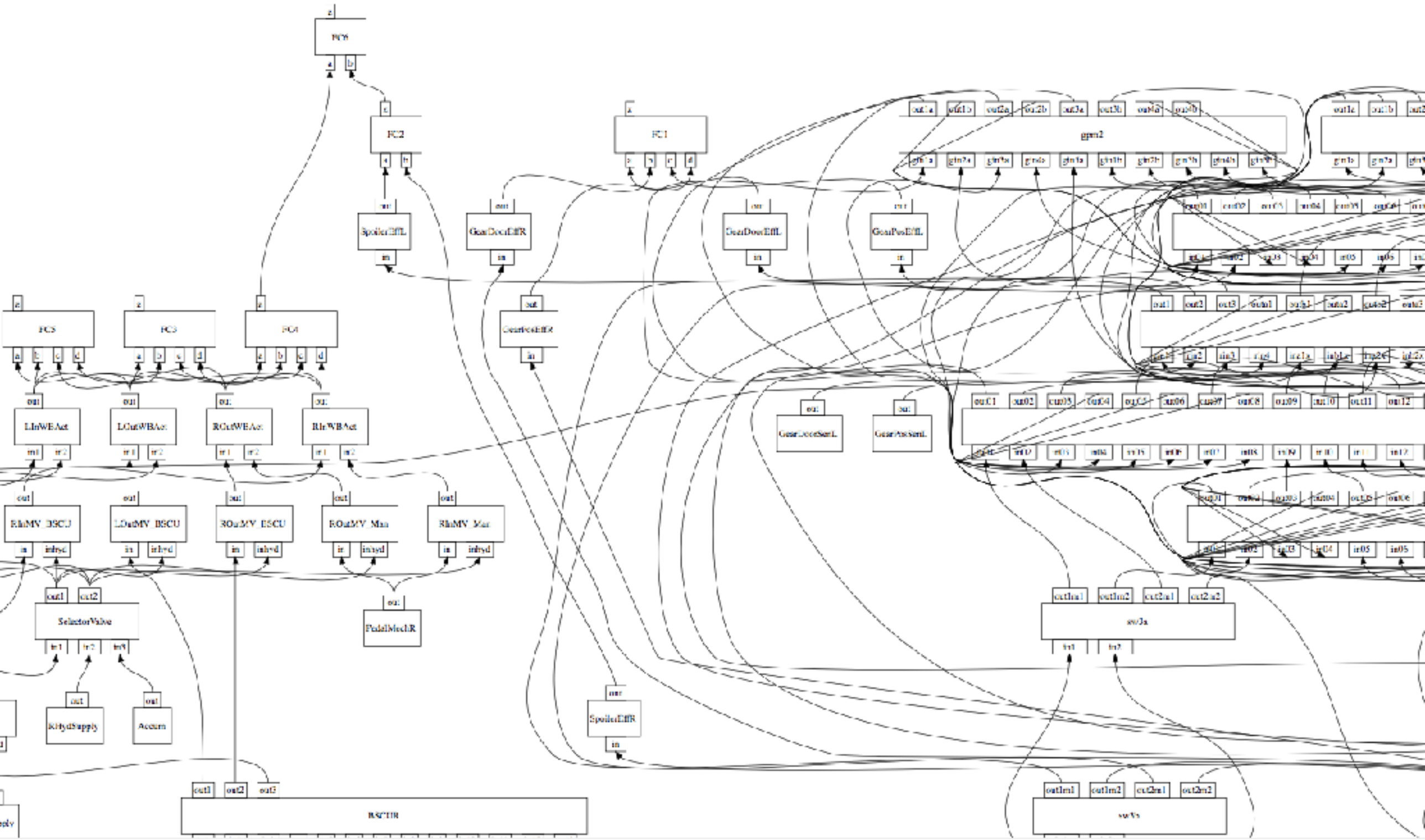
Functional Architecture



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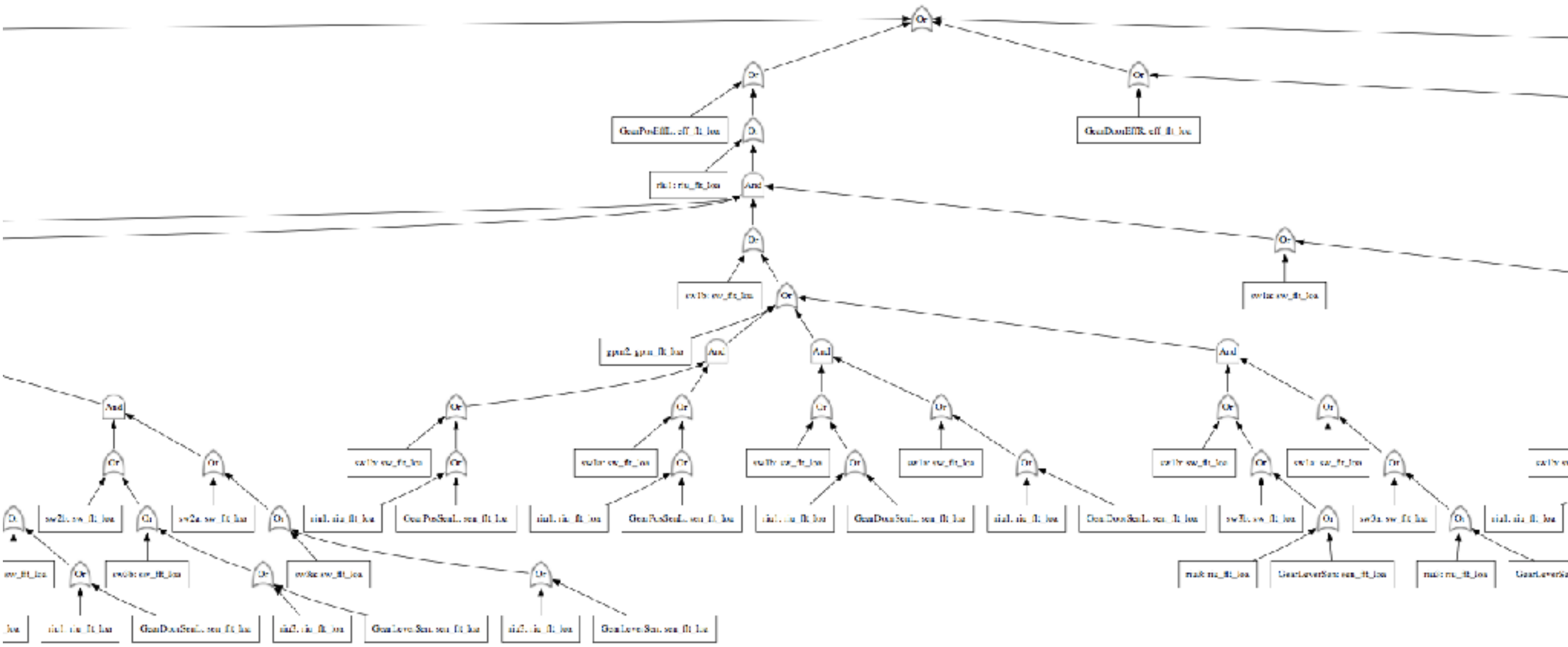
Fault Tree Synthesized

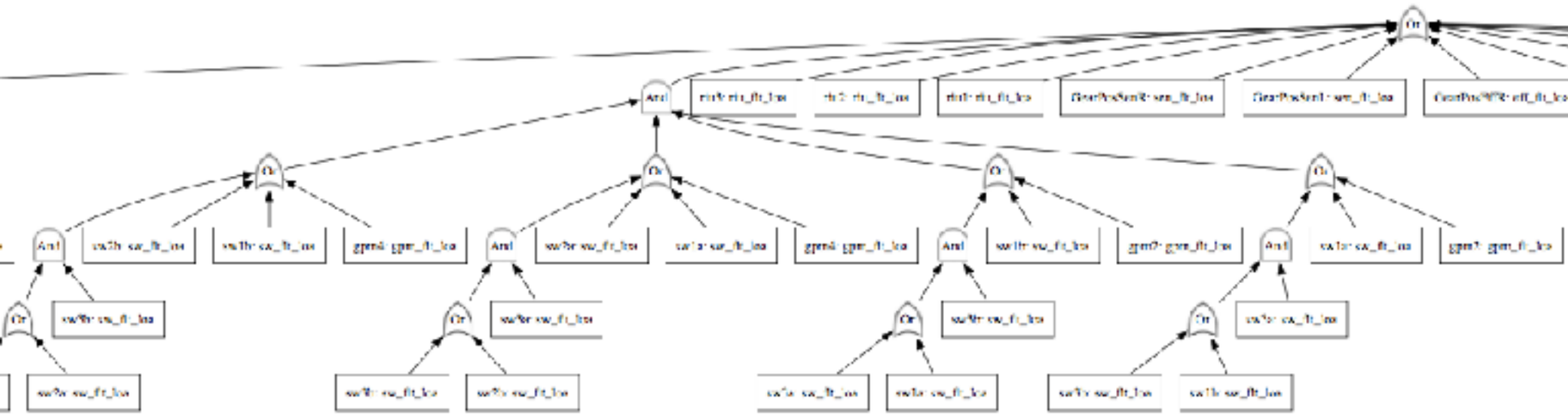


Large fault tree: 549 nodes, 261 Gates, Depth 11

The problem: safety engineer has to perform mental gymnastics to understand

The challenge: can we generate fault trees that a human might generate?

Fault Tree Reduction Engine



Nodes: 549 ➔ 83, Gates: 261 ➔ 27, Depth: Depth 11 ➔ 5

Notes: reduction engine does not increase depth

returns semantically equivalent tree

yields significant simplification of fault tree

fault trees now look like something a human generated