Lecture 14

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Project Presentations

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Skolem Normal Form Example

For any FO ϕ, we can find a universal ψ in an *expanded* language such that ϕ is satisfiable iff ψ is satisfiable. Try it!

$$\langle \exists x \ \langle \forall w \ \langle \exists y \ \langle \forall u, v \ \langle \exists z \ \phi(x, w, y, u, v, z) \rangle \rangle \rangle \rangle \rangle$$

First, PNF, and push existentials left (2nd order logic)

$$\langle \exists x, F_y \ \langle \forall w, u, v \ \langle \exists z \ \phi(x, w, F_y(w), u, v, z) \rangle \rangle \rangle$$

$$\langle \exists x, F_y, F_z \ \langle \forall w, u, v \ \phi(x, w, F_y(w), u, v, F_z(w, u, v)) \rangle \rangle$$

The key idea is the following equivalence

We need the axiom of choice for ping

$$\langle \exists \ldots \ \langle \forall x_1, \ldots x_n \ \langle \exists y \ \phi(\ldots, x_1, \ldots, x_n, y) \rangle \rangle \rangle$$

$$\equiv \ \langle \exists \ldots \ \langle \exists F_y \ \langle \forall x_1, \ldots, x_n \ \phi(\ldots, x_1, \ldots, x_n, F_y(x_1, \ldots, x_n)) \rangle \rangle \rangle$$

This allows us to push existential quantifiers to the left

To get back to FO, note that

Sat$\langle \exists \ldots \ \langle \forall x_1, \ldots x_n \ \langle \exists y \ \phi(\ldots, x_1, \ldots, x_n, y) \rangle \rangle \rangle$ **iff**

Sat$\langle \forall x_1, \ldots, x_n \ \phi(\ldots, x_1, \ldots, x_n, F_y(x_1, \ldots, x_n)) \rangle$

So, to finish our example, we get, where *c*, F_y, F_z are new symbols,

$$\langle \forall w, u, v \ \phi(c, w, F_y(w), u, v, F_z(w, u, v)) \rangle$$

Skolem Normal Form Algorithm

Convert formula to NNF

Notice that Skolemizing in arbitrary formulas doesn't work (hence NNF)

$\langle \exists x\ P(x) \rangle \wedge \neg \langle \exists y\ P(y) \rangle$ is not equisatisfiable with $\langle \exists x\ P(x) \rangle \wedge \neg P(d)$

is equisatisfiable with $P(c) \wedge \langle \forall y \neg P(y) \rangle$

Only works with positive polarity formulas, which NNF guarantees

With NNF, we can apply Skolemization to any sub formula

$\langle \forall x, z\ x = z \vee \langle \exists y\ x \cdot y = 1 \rangle \rangle$ can be Skolemized as

$\langle \forall x, z\ x = z \vee x \cdot f(x) = 1 \rangle$ or we can convert to PNF

$\langle \forall x, z\ \langle \exists y\ x = z \vee x \cdot y = 1 \rangle \rangle$ and then Skolemize

$\langle \forall x, z\ x = z \vee x \cdot f(x, z) = 1 \rangle$ *order matters!*

So, it is better to Skolemize inside-out and then convert to PNF

FO Sat/Validity Reductions

Theorem: For any FO ϕ, we can find a universal ψ in an *expanded* language such that ϕ is satisfiable iff ψ is satisfiable. (Proof in previous slide)

Previous example

$$\langle \exists x \langle \forall w \langle \exists y \langle \forall u, v \langle \exists z\ \phi(x, w, y, u, v, z)\rangle\rangle\rangle\rangle\rangle$$

$$\langle \forall w, u, v\ \phi(c, w, F_y(w), u, v, F_z(w, u, v))\rangle$$

Notice that our approach does not give an equi-valid formula. Consider:

$$\langle \forall x \langle \exists y\ P(x) \Rightarrow P(y)\rangle\rangle$$

$$\langle \forall x\ P(x) \Rightarrow P(f_y(x))\rangle$$

Both formulas are satisfiable; the first is valid but the second is not

Corollary: For any FO ϕ, we can find an existential ψ in an *expanded* language such that ϕ is valid iff ψ is valid

Pf: ϕ is valid iff ¬ϕ is unsat iff (universal) ϕ' is unsat iff (existential) ψ=¬ϕ' is valid

$$\phi = \langle \forall x \langle \exists y\ P(x) \Rightarrow P(y)\rangle\rangle \quad \rightarrow \quad \neg\phi = \langle \exists x \langle \forall y\ P(x) \wedge \neg P(y)\rangle\rangle$$

$$\phi' = \langle \forall y\ P(c) \wedge \neg P(y)\rangle \quad \rightarrow \quad \psi = \langle \exists y\ P(c) \Rightarrow P(y)\rangle$$

So FO Sat reduced to FO universal Sat and FO Validity to FO universal Unsat

Reduction to Propositional SAT

- We reduced FOL SAT to SAT of the universal fragment
- We now go one step further

ground: quantifier/variable free

- Theorem: A universal FO formula (w/out =) is SAT iff all finite sets of ground instances are (propositionally) SAT (eg $P(x) \vee \neg P(x)$ is propositionally SAT)
- Corollary: A universal FO formula (w/out =) is UNSAT iff some finite set of ground instances is (propositionally) UNSAT
- FO validity checker: Given FO ϕ, negate & Skolemize to get universal ψ s.t. Valid(ϕ) iff UNSAT(ψ). Let G be the set of ground instances of ψ (possibly infinite, but countable). Let $G_1, G_2 \ldots,$ be a sequence of subsets of G s.t. $\forall g \subseteq G, \exists n$ s.t. $g \subseteq G_n$. If $\exists n$ s.t. Unsat G_n, then Unsat ψ and Valid ϕ.
- The SAT checking is done via a propositional SAT solver!
- If ϕ is not valid, the checker may never terminate, i.e., we have a semi-decision procedure and we'll see that's all we can hope for
- How should we generate G_i? One idea is to generate all instances over terms with at most 0, 1, … , functions. We'll explore that more later.

Example

$\langle \exists x \, \langle \forall y \; P(x) \Rightarrow P(y) \rangle \rangle$ **is Valid iff** $\langle \forall x \, \langle \exists y \; P(x) \wedge \neg P(y) \rangle \rangle$ **is UNSAT**

iff $\langle \forall x \; P(x) \wedge \neg P(f_y(x)) \rangle$ **is UNSAT**

- *Herbrand universe* of FO language L is the set of all ground terms of L, except that if L has no constants, we add c to make the universe non-empty.
- For our example we have $H = \{c, f_y(c), f_y(f_y(c)), \ldots\}$
- So $G = \{P(t) \wedge \neg P(f_y(t)) \mid t \in H\}$
- Notice that $\Delta = \{P(c) \wedge \neg P(f_y(c)), P(f_y(c)) \wedge \neg P(f_y(f_y(c)))\}$ is UNSAT
  - the SAT solver will report UNSAT for: $P(c) \wedge \neg P(f_y(c)) \wedge P(f_y(c)) \wedge \neg P(f_y(f_y(c)))$
- So, for the first G_i that has both $\neg P(f_y(c))$ and $P(f_y(c))$ will lead to termination