Syntax of FOL

Definition 1 *$\mathcal{A}$ contains the following symbols:*

1. $v_0, v_1, v_2, \ldots$ *(variables);*

2. $\neg, \wedge, \vee, \Rightarrow, \leftrightarrow$ *(boolean connectives);*

3. $\forall, \exists$ *(quantifiers);*

4. $\equiv$ *(equality symbol);*

5. $), ($ *(parenthesis);*

There may also be other symbols in a FOL, *e.g.*, in set theory we have $\in$, a 2-ary relation symbol.

Definition 2 *The symbol set S of a FOL contains*

1. *for every $n \geq 1$ a (possibly empty) set of n-ary relation symbols.*
2. *for every $n \geq 1$ a (possibly empty) set of n-ary function symbols.*
3. *a (possibly empty) set of constant symbols.*

$\mathcal{A}_S := \mathcal{A} \cup S$ is the alphabet of this language.

We use $P, Q, R, \ldots$ for relation symbols, $f, g, h, \ldots$ for function symbols, $c, c_0, c_1, \ldots$ for constants, and $x, y, z, \ldots$ for variables.

Terms

Definition 3 *The set of S-terms, denoted T^S is the least set closed under the following rules.*

1. Every variable is an S-term.

2. Every constant in S is an S-term.

3. If $t_1, \ldots, t_n$ are S-terms and f is an n-ary function symbol in S, then $ft_1 \ldots t_n$ is an S-term.

Note that $T^S \subseteq \mathcal{A}_S^*$.

Here is an analogy with English. Bill, the father of John, etc. all denote elements in our universe. Similarly, x, c, fxy, etc. denote elements of the universe of a first-order theory.

Formulas

Terms name objects in our domain, whereas formulas correspond to statements about our domain.

Statements such as "Bob has three siblings" are statements about the universe. They are either true or false. That is the role played by formulas.

Definition 4 *An* atomic formula *of S is either of the form $t_1 \equiv t_2$ or $Rt_1 \ldots t_n$, where $t_1, t_2, \ldots, t_n$ are S-terms and R is an n-ary relation symbol in S.*

Definition 5 *The set of S-*formulas *is the least set closed under the following rules.*

1. Every atomic formula is an S-formula.

2. If φ, ψ are S-formulas and x is a variable, then $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, $\exists x\varphi$, and $\forall x\varphi$ are S-formulas.

L^S denotes the set of S-formulas.

Are the following in $L^{S_{gr}}$?

- $\forall v_0 \circ e v_0 \equiv e$
- $\forall v_1 \circ e v_0 \equiv e$
- $\forall v_1 \circ e v_0 \equiv e \quad \wedge \quad \forall v_1 \circ e v_0 \equiv e$
- $\forall v_1 \exists v_1 \circ v_1 v_1 \equiv v_1$
- $\forall v_1 \exists v_1 (\circ v_1 v_1 \rightarrow v_1)$

Is there a string that is both a formula and a term?

Can you think of a formula that can be parsed in more than one way?

Lemma 1 *If* $|S| \leq \omega$, *then* $|T^S| = |L^S| = \omega$.

Definitions on terms and formulas

Define a function that given an S-term returns the set of variables occuring in it.

Formulas without free variables are called *sentences*. Define a function that given an S-formula returns the set of free variables occuring in it.

Semantics of FOL

We will now go beyond the grammatical, syntactic aspects of FOL to discuss what terms and formulas mean. Notions such as *free*, *term*, *formula* are purely syntactic.

Here is an example of something that isn't syntactic: what does $\forall v_0 R v_0 v_1$ mean? Well, it depends on what R means, *i.e.*, what relation is it and over what domain? and what v_1 means, *i.e.*, what element of the domain is it? Say that R is $<$ on $\mathbb{N}$ and v_1 is 0, then the statement is false. If R is $\geq$, then it is true.

Structures

Definition 6 *An* S-structure *is a pair* $\mathbf{U} = \langle A, \mathbf{a} \rangle$*, where* A *is a non-empty set, the* domain *or* universe*, and* $\mathbf{a}$ *is a function with domain* S *such that:*

1. If $c \in S$ is a constant symbol, then $\mathbf{a}.c \in A$.

2. If $f \in S$ is an n-ary function symbol, then $\mathbf{a}.f : A^n \rightarrow A$.

3. If $R \in S$ is an n-ary relation symbol, then $\mathbf{a}.R \subseteq A^n$.

Instead of $\mathbf{a}.R, \mathbf{a}.f$, and $\mathbf{a}.c$ we often write $R^{\mathbf{U}}$, $f^{\mathbf{U}}$, and $c^{\mathbf{U}}$ or even R^A, f^A, and c^A.

Instead of denoting a structure $\mathbf{U}$ as a pair $\langle A, \mathbf{a} \rangle$, we often replace $\mathbf{a}$ by a list of its values, *e.g.*, we would write an $\{f, R, c\}$-structure as $\langle A, f^{\mathbf{U}}, R^{\mathbf{U}}, c^{\mathbf{U}} \rangle$.

Here are some examples. The symbol sets

$$S_{ar} := \{+, \cdot, 0, 1\} \text{ and } S_{ar}^{<} := \{+, \cdot, 0, 1, <\}$$

play an important role, and we use $\mathcal{N}$ to denote the S_{ar}-structure $\langle \mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}} \rangle$ and $\mathcal{N}^{<}$ to denote the $S_{ar}^{<}$-structure $\langle \mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}} \rangle$.

Similarly, we use $\mathcal{R}$ to denote the S_{ar}-structure $\langle \mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}} \rangle$ and $\mathcal{R}^{<}$ to denote the $S_{ar}^{<}$-structure $\langle \mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, <^{\mathbb{R}} \rangle$.

Notice that $+^{\mathbb{R}}$ and $+^{\mathbb{N}}$ are very different objects. Even so, we will drop the subscripts when (we think) no ambiguity will arise.