Proof Theory

We will define $\Phi \vdash \varphi$: φ is provable from Φ.

A *sequent* is a nonempty list (sequence) of formulas. For example, $\varphi_1 \ldots \varphi_n \varphi$ is a sequent. $\varphi_1 \ldots \varphi_n$ is called the *antecedent* and φ is the *succedent*.

We will use $\Gamma, \Delta, \ldots$ to denote (possibly empty) sequences of formulas. We will now define a sequent calculus. Here is an example.

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\Gamma \quad \varphi}$$

Think of this as saying that if you have a proof of both ψ and $\neg\psi$ from $\Gamma \cup \{\neg\varphi\}$ then that constitutes a proof of φ from Γ.

Derivability

If there is a derivation of the sequent $\Gamma\ \varphi$, then we write $\vdash \Gamma\ \varphi$ and we say that $\Gamma\ \varphi$ is *derivable*.

Definition 1 *A formula φ is* formally provable *or* derivable *from a set Φ of formulas (written $\Phi \vdash \varphi$) iff there are finetely many formulas $\varphi_1, \ldots, \varphi_n$ in Φ such that $\vdash \varphi_1\ \ldots\ \varphi_n\ \varphi$.*

A sequent $\Gamma\ \varphi$ is *correct* if $\Gamma \models \varphi$

Sequent Calculus

Antecedent Rule (Ant)

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}$$ if every member of Γ is also a member of Γ'.

Assumption Rule (Assm)

$$\frac{}{\Gamma \quad \varphi}$$ if φ is a member of Γ

Proof by Cases Rule (PC)

$$\frac{\begin{array}{l}\Gamma \quad \psi \quad \varphi \\ \Gamma \quad \neg\psi \quad \varphi\end{array}}{\Gamma \quad\quad \varphi}$$

Sequent Calculus

Contradiction Rule (Ctr)

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\Gamma \quad \varphi}$$

$\vee$-Rule for the Antecedent ($\vee$ A)

$$\frac{\begin{array}{lll}\Gamma & \varphi & \xi \\ \Gamma & \psi & \xi\end{array}}{\Gamma \quad (\varphi \vee \psi) \quad \xi}$$

$\vee$-Rule for the Succedent ($\vee$ S)

(a) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)}$ (b) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$

Derived Rules

Using the existing rules, we can derive new sequents.

Tertium non datur (Ctr)

$$\frac{}{(\varphi \vee \neg\varphi)}$$

Proof? We can prove it by assuming φ, getting $\varphi \vee \neg\varphi$ and repeat with $\neg\varphi$.

Second Contradiction Rule (Ctr')

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

More Derived Rules

Chain Rule (Ch)

$$\frac{\begin{array}{lll}\Gamma & & \varphi \\ \Gamma & \varphi & \psi\end{array}}{\begin{array}{lll}\Gamma & & \psi\end{array}}$$

Contraposition Rules (Cp)

(a) $$\frac{\Gamma \quad \varphi \quad \psi}{\Gamma \quad \neg\psi \quad \neg\varphi}$$

(c) $$\frac{\Gamma \quad \neg\varphi \quad \psi}{\Gamma \quad \neg\psi \quad \varphi}$$

(b) $$\frac{\Gamma \quad \neg\varphi \quad \neg\psi}{\Gamma \quad \psi \quad \varphi}$$

(d) $$\frac{\Gamma \quad \varphi \quad \neg\psi}{\Gamma \quad \psi \quad \neg\varphi}$$

Modus ponens

$$\frac{\begin{array}{ll}\Gamma & (\varphi \rightarrow \psi) \\ \Gamma & \varphi\end{array}}{\begin{array}{ll}\Gamma & \psi\end{array}}$$

Quantifier Rules

$\exists$-Introduction in the Succedent ($\exists$ S)

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x \varphi}$$

Proof

The next rule corresponds to an often used argument used to prove that ψ follows from $\exists x \varphi$. One assumes that for some new y, $\varphi\frac{y}{x}$.

$\exists$-Introduction in the Antecedent ($\exists$ A)

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x \varphi \quad \psi} \text{ if } y \text{ is not free in } \Gamma \ \exists x \varphi \ \psi.$$

Proof

Equality Rules

Reflexivity Rule for Equality ($\equiv$)

$$\frac{}{t \equiv t}$$

Substitution Rule for Equality (Sub)

$$\frac{\Gamma \qquad\qquad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

Soundness of the Sequent Calculus

Recall: a formula φ is derivable from Φ, written $\Phi \vdash \varphi$, iff there are formulas $\varphi_1, \ldots, \varphi_n$ in Φ such that $\vdash \varphi_1 \ldots \varphi_n \varphi$.

Lemma 1 *For all Φ and φ, $\Phi \vdash \varphi$ iff there is a finite subset Φ_0 of Φ such that $\Phi_0 \vdash \varphi$.*

We will prove a similar theorem, the compactness theorem, for $\models$. As a preview, once we prove the completeness theorem, namely that the notions $\models$ and $\vdash$ are "equivalent" then we will be able to transfer results such as this one from one realm to the other. The beauty is that sometimes results are trivial to prove in one realm, but seem very deep in the other.

Theorem 1 *For all Φ and φ, if $\Phi \vdash \varphi$ then $\Phi \models \varphi$.*

Proof The proof is by induction on the structure of a derivation.

This is one direction of the completeness theorem.