Graph Algorithms

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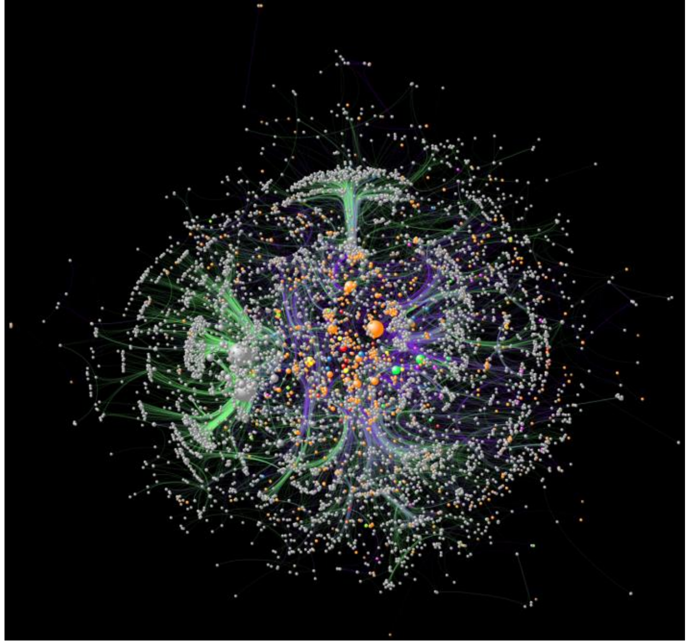
Key Learning Goals

- Write the pseudo-code for breadth-first search (BFS) in MapReduce and in Spark.
- Write the pseudo-code for single-source shortest path in MapReduce and in Spark.
- Write the pseudo-code for PageRank (without dangling pages) in MapReduce and in Spark.
- How can the single-source shortest path algorithm detect that no more iterations are needed?
- How can the PageRank algorithm detect that no more iterations are needed?

Key Learning Goals

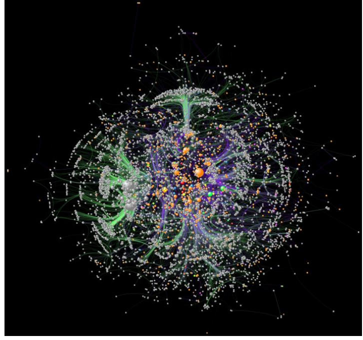
- Why is Spark better suited for BFS and BFS-based algorithms than MapReduce?
- How can we handle dangling pages in PageRank?

Introduction



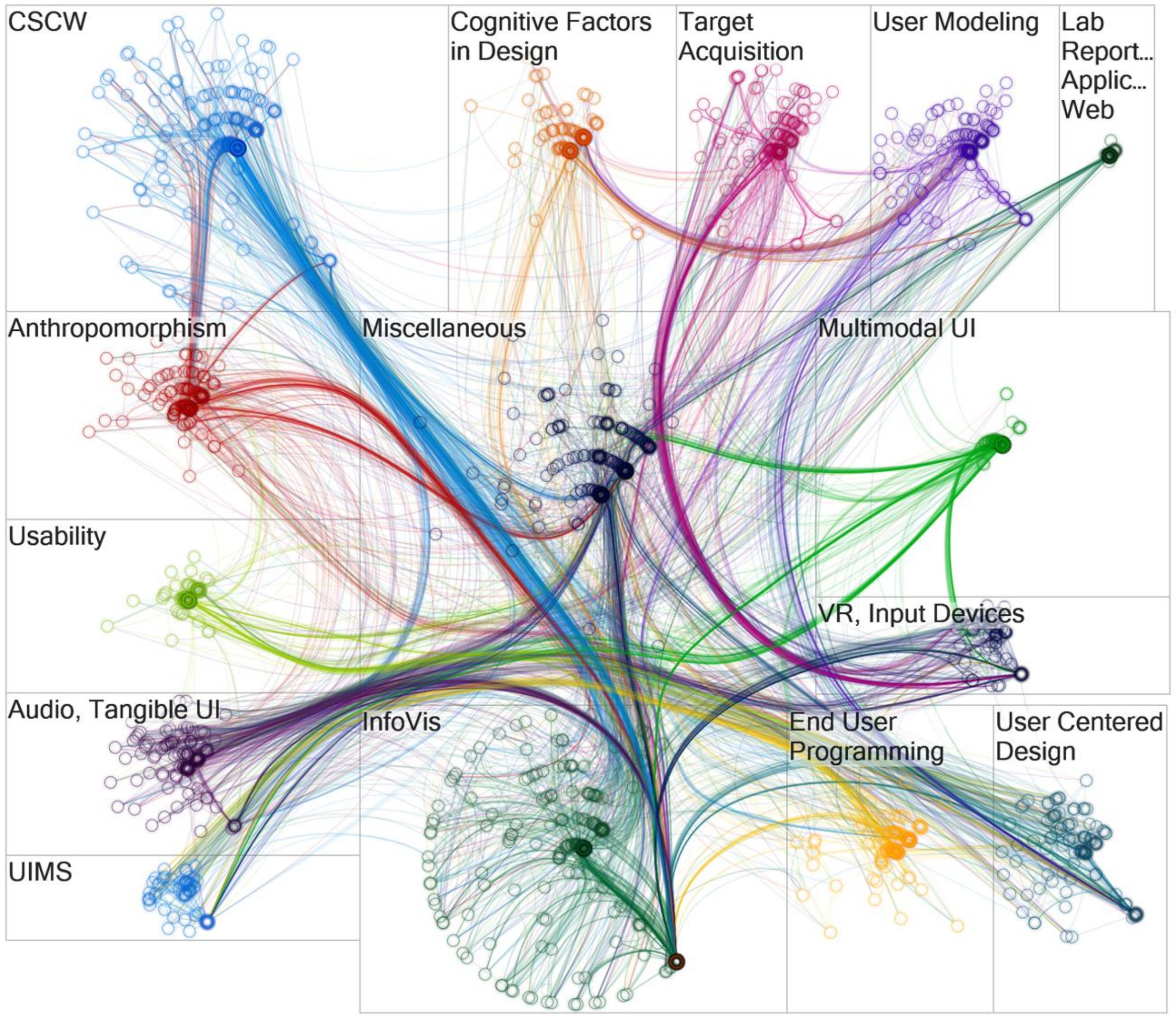
- Graphs are very general and a large variety of real-world problems can be modeled naturally as graph analysis or mining problems. Data occurring as graphs include:
  - The hyperlink structure of the Web
  - Social interactions, e.g., Facebook friendships, Twitter followers, email flows, and phone call patterns
  - Transportation networks, e.g., roads, bus routes, and flights
  - Relationships between genes, proteins, and diseases
- Since graphs are so general, many graph problems are inherently complex—a perfect target for distributed data-intensive computation.

Explanation for the Example

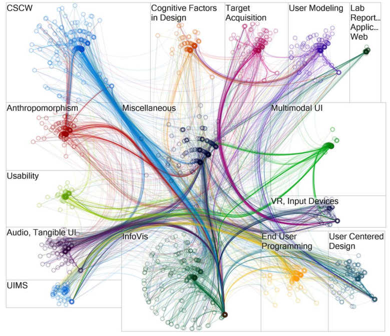


- This image shows a network of innovation relationships in the state of Pennsylvania in 1990, collected by Christopher S. Dempwolf. The node-link visualization is laid out with the Harel-Koren FMS layout algorithm. Orange nodes are companies, e.g., Westinghouse electric; silver nodes inventors. Size indicates a measure of prominence. Edge colors represent the type of a collaboration and edges with similar routes are bundled together.
  - Chaturvedi, S, Dunne, C, Ashktorab, Z, Zacharia, R, and Shneiderman, B. “Group-in-a-Box meta-layouts for topological clusters and attribute-based groups: space efficient visualizations of network communities and their ties”. In: CGF: Computer Graphics Forum 33.8 (2014), pp. 52-68
  - Note: Cody Dunne is a faculty member in our college.

Relationships between CHI Topics



Explanation for the Example



- Papers of the ACM CHI conference, grouped disjointly by the main topic they cover and laid out individually for each topic using a polar layout. Node colors show the various groups, and the color of edges shows citations from papers in that same colored group to other papers. Edges with similar routes are bundled together. Radius and angle encode the number of citations and the betweenness centrality (how much of a gatekeeper the paper is between groups).
  - Same reference as previous image.

What is a Graph?

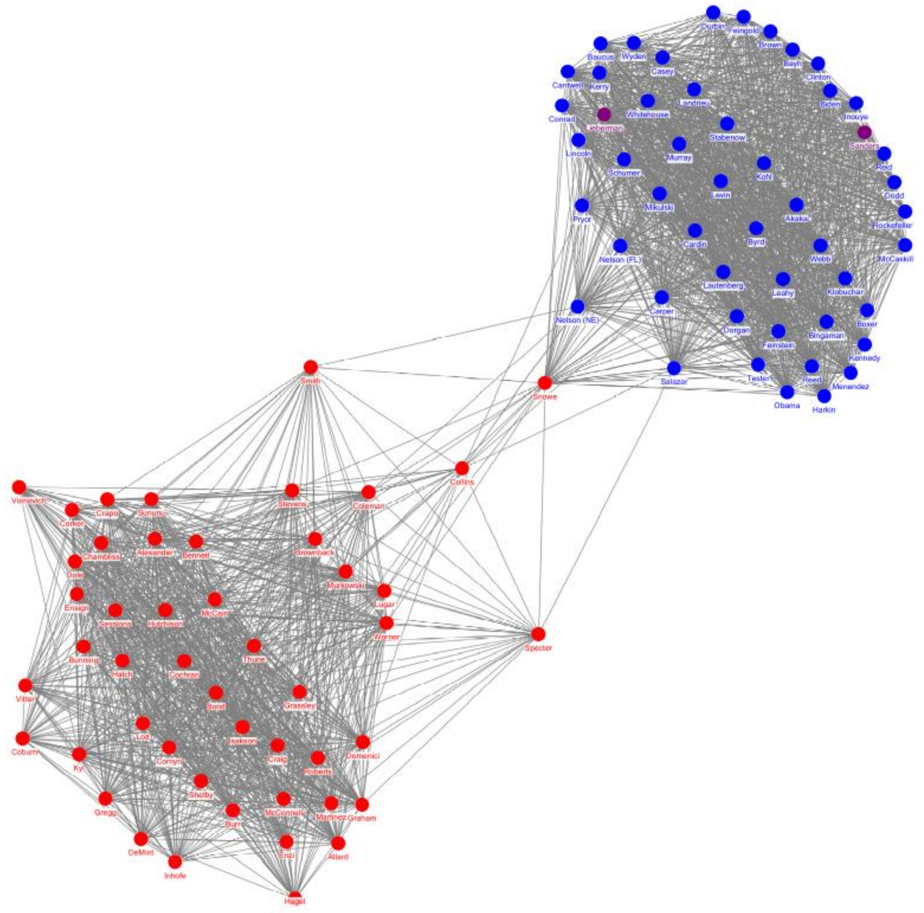
- A graph consists of vertices (a.k.a. nodes) and edges (a.k.a. links) between them. Vertex and edge labels can be used to encode heterogeneous networks.
  - In an online social network, a vertex could be a person annotated with demographic information such as age. An edge could be annotated by the type of relationship, e.g., “friend” or “family.” This richer structure can improve the quality of the results obtained from graph analysis.
- We will focus on unlabeled graphs. Formally, a graph G is a pair (V, E), where V is a set of vertices and E is set of edges, such that $E \subseteq V \times V$.
- Edges can be directed or undirected. In a directed graph, edges (v1, v2) and (v2, v1) are different, while in an undirected graph they are the same. A standard trick is to encode undirected edge (v1, v2) using the corresponding two directed edges.
  - A road network should be modeled as a directed graph, because of the existence of one-way streets.
- Graphs might contain cycles. If a graph does not contain any cycle, it is acyclic.

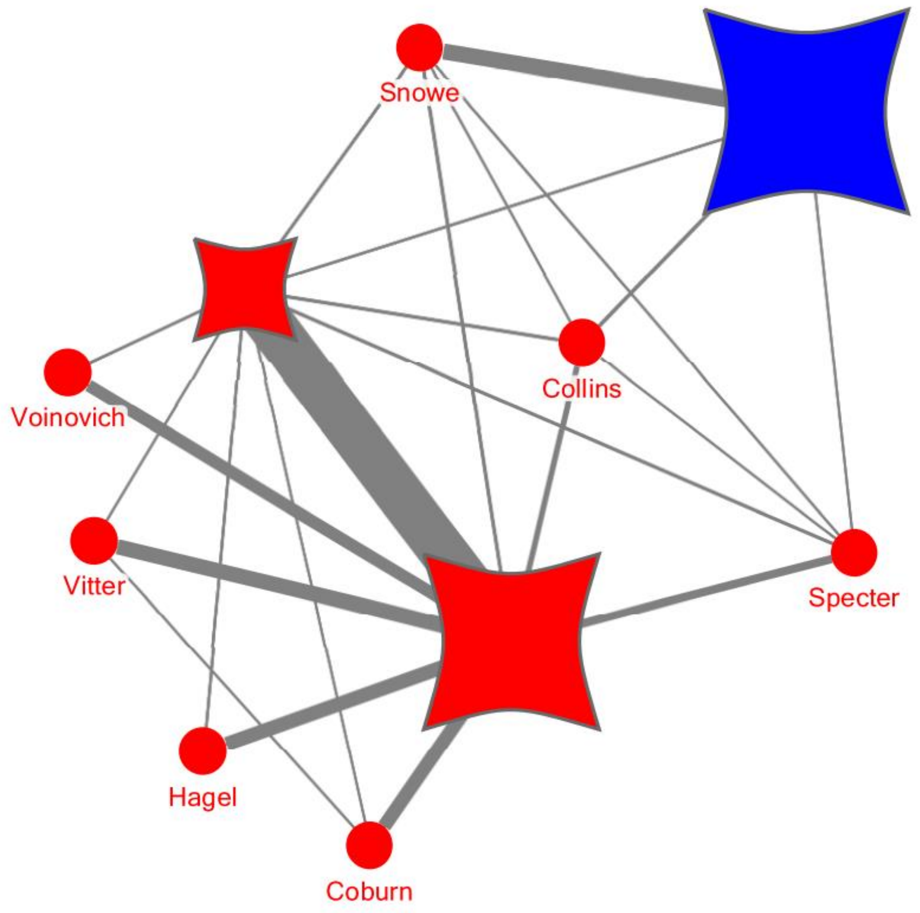
Graph Problems

- There is a huge variety of graph analysis problems, including the following common examples.

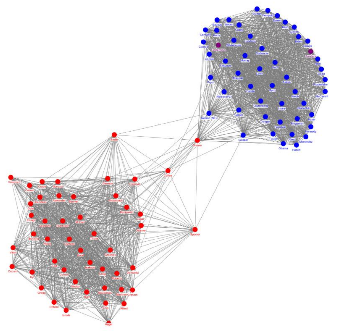
- Graph search and path planning
  - Find driving directions from point A to point B.
  - Recommend new friends in a social network.
  - Find the best route for IP packets or delivery trucks.
- Graph clustering
  - Identify user communities in social networks based on graph structure, e.g., connectedness.
  - Partition a large graph into homogeneous partitions to parallelize graph processing.
- Minimum spanning trees
  - Find the connected graph of minimum total edge weight.
- Bipartite graph matching
  - Match nodes on the “left” with nodes on “right” side. For example, match job seekers and employers, singles looking for dates, or papers with qualified reviewers.
- Maximum flow
  - Determine the greatest possible traffic between a source and a sink node, e.g., to optimize transportation networks.
- Finding “special” vertices
  - Find vertices that are of importance, e.g., disease hubs, leaders of a community, authoritative Web pages on a topic, or people with influence.

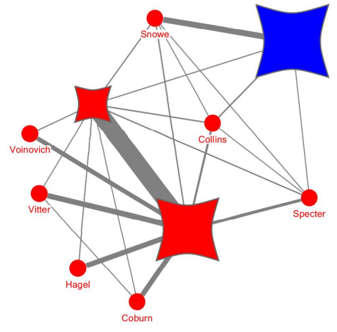
US Senate Co-Voting Patterns





Explanation for the Example





- US Senate co-voting patterns in 2007. Nodes represent senators and are colored by party: blue democrats, red republicans, purple independents. Edges join pairs of senators that vote the same way at least 70% of the time. The left image shows a node-link visualization of the data arranged using the Fruchterman Reingold network layout algorithm. The right shows the same network after applying motif simplification, which aggregates maximal cliques in the network into tapered square glyphs. In this the size of edges shows the overall number of edges between any pair of glyphs or glyphs and nodes.
  – Dunne, C and Shneiderman, B. “Motif simplification: improving network visualization readability with fan, connector, and clique glyphs”. In: CHI '13: Proc. SIGCHI Conference on Human Factors in Computing Systems. 2013, pp. 3247-3256

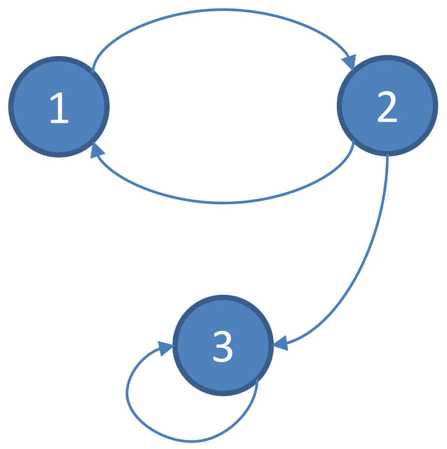
Graph Representations

- Graphs are usually represented in one of these three formats:
  - Adjacency matrix
  - Adjacency list
  - Set of individual edges
- We will take a quick look at each of them.

Adjacency-Matrix Representation

- The adjacency matrix M represents a graph in a matrix of size |V| by |V|. Its size is quadratic in the number of vertices. Entry M(i,j) contains the weight of the edge from vertex i to vertex j; or 0 if there is no edge.
  - The adjacency matrix of an undirected graph is symmetric along the diagonal.

	1	**2**	**3**
1	0	1	0
2	1	0	1
3	0	0	1



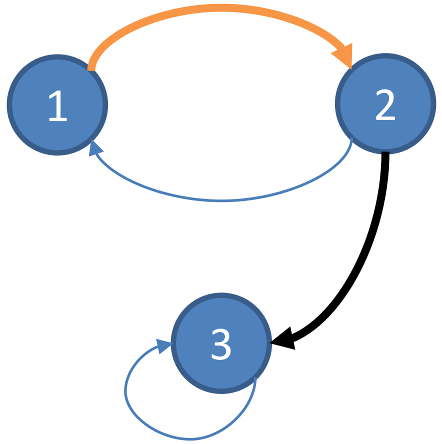
Adjacency Matrix Properties

- Pro: An adjacency matrix is easy to process using linear algebra. For example, in product M×M, entry (i,j) indicates if there exists a two-step path from vertex i to vertex j (if the value is non-zero) or not (if the value is zero).
- Pro: An operation on outgoing edges of a vertex (also called "outlinks") corresponds to an iteration over a row of the adjacency matrix. An operation on incoming edges of a vertex (also called "inlinks") corresponds to an iteration over a column.
- Con: For large graphs, the adjacency matrix tends to be very sparse as most vertex pairs are not connected by an edge. For those graphs this representation is inefficient (in terms of storage cost) or even infeasible. Consider the Facebook friendship graph: for 1 billion users, even if every user on average had 10,000 friends, still 99.999% of the matrix would have value zero.

Two-Hop Paths in Linear Algebra

- Consider entry (1, 3) in matrix M×M, which is marked in green. Its value 1 is caused by M(1, 2)=1 (marked orange) and M(2, 3)=1 (marked black). Intuitively, the two-step path from node 1 to node 3 is made up of edges (1, 2) and (1, 3).

$$\underset{M}{\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}} \times \underset{M}{\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & \mathbf{1} \\ \hline 0 & 0 & 1 \\ \hline \end{array}} = \underset{M\times M}{\begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}}$$



Adjacency-List Representation

- The adjacency-list representation only stores the existing *outlinks* for each vertex. Hence its size is linear in the number of edges.

Adjacency matrix

	1	**2**	**3**
1	0	1	0
2	1	0	1
3	0	0	1

Adjacency list for the same graph

1: 2
2: 1, 3
3: 3

Adjacency-List Properties

- Pro: Compared to the adjacency matrix, the adjacency list is much more space-efficient for sparse graphs.
- Pro: It is still easy to compute over the outlinks of a vertex, because they are stored in the adjacency list for that vertex.
- Con: Computation over the inlinks of a vertex is more challenging compared to the adjacency matrix. Instead of scanning through a column, each of the different lists must be searched, e.g., using binary search if the lists are sorted.

Set-of-Edges Representation

- The set-of-edges approach stores each edge of the graph explicitly.
  - The adjacency list format is set-of-edges, grouped by the "start" vertex of each edge.

Adjacency matrix

	1	**2**	**3**
1	0	1	0
2	1	0	1
3	0	0	1

Set of edges for the same graph

(1,2), (2,1), (2,3), (3,3)

Set-of-Edges Properties

- Pro: This representation is still space-efficient for sparse graphs and it is easy to perform per-edge manipulations, e.g., inverting each edge.
- Pro: The uniform record format—each record is a pair of vertices, in contrast to adjacency lists, whose sizes can vary—can simplify programming and data processing.
- Con: Out of the three, this is the most challenging format for computations over inlinks or outlinks of a vertex. The records corresponding to a given vertex' inlinks and outlinks might be scattered all over the file storing the set of edges.

Let us first look at the classic breadth-first algorithm for exploring the k-hop neighborhood of a given source vertex.

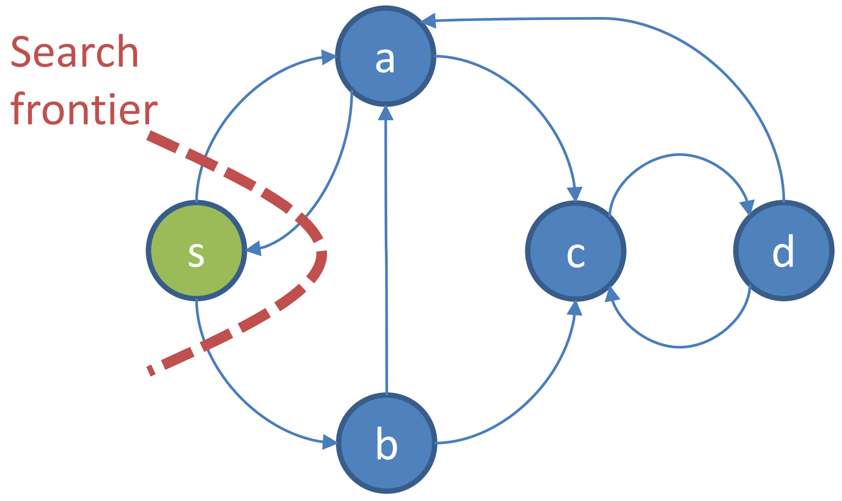
This algorithm forms the basis of two others we discuss later.

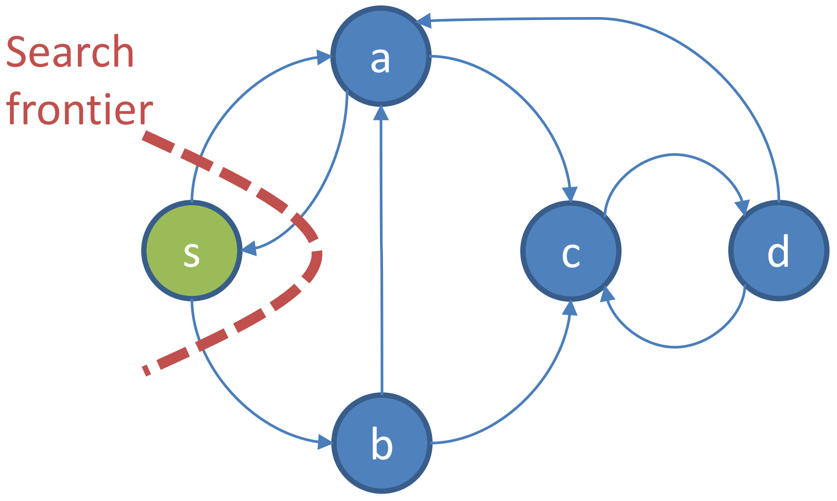
Breadth-First Search (BFS)

- Starting from a given source vertex s, breadth-first search first reaches all 1-hop neighbors of s, then all 2-hop neighbors, and so on.
- A vertex will be encountered repeatedly if and only if it can be reached through different paths.
- If the graph has cycles, then the search process can continue indefinitely, repeatedly traversing the cycles.
- In an acyclic graph, breadth-first search will terminate after at most |V|-1 iterations—this is the length of the longest path possible in any acyclic graph.

BFS Example

- The following images illustrate how breadth-first traversal reaches vertices in each iteration.
  - Green color highlights all vertices visited in the corresponding iteration. Notice how some vertices are visited repeatedly.
  - Due to the particular structure of the example graph, after iteration 3, each vertex will be reached in each of the following iterations.
- The search frontier illustrates how breadth-first traversal reaches new vertices.
  - Iteration 1: Vertices a and b are reached in exactly one hop from s.
  - Iteration 2: Vertex s is reached again (from a), and so is a (from b). Vertex c is reached for the first time.
  - Iteration 3: All vertices are reached, including d for the first time.


Search frontier
a
s
c
d
b


Search frontier
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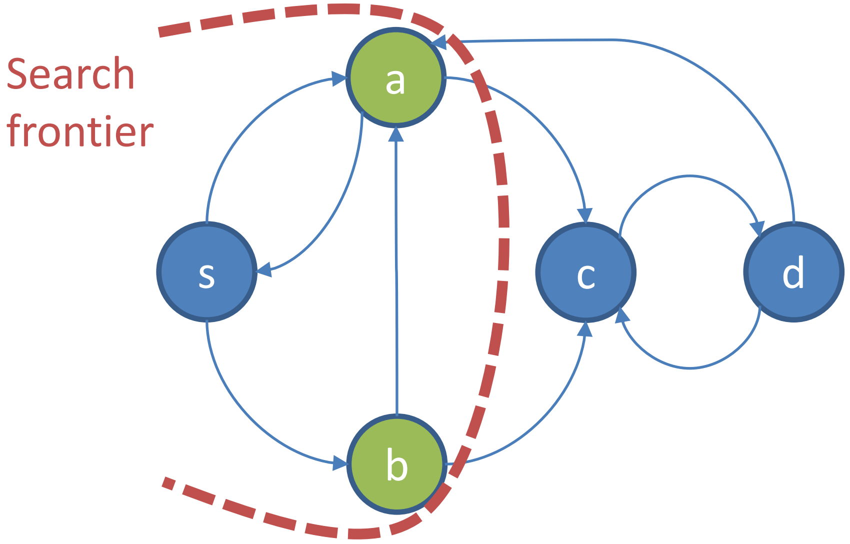
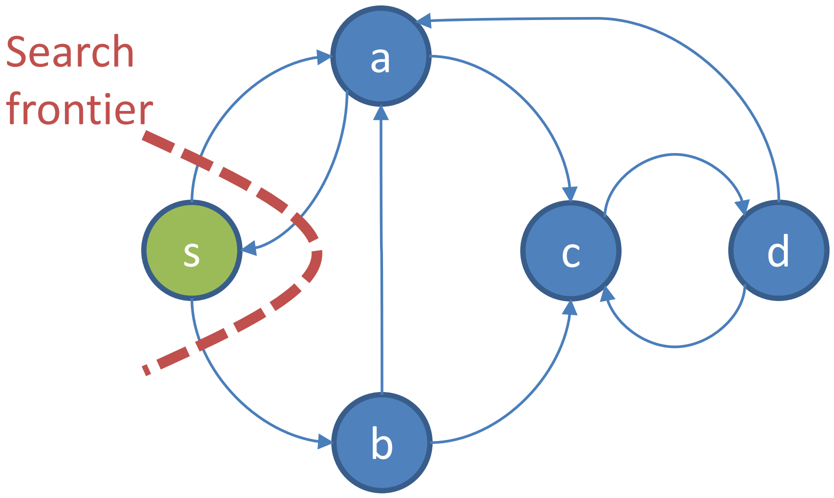
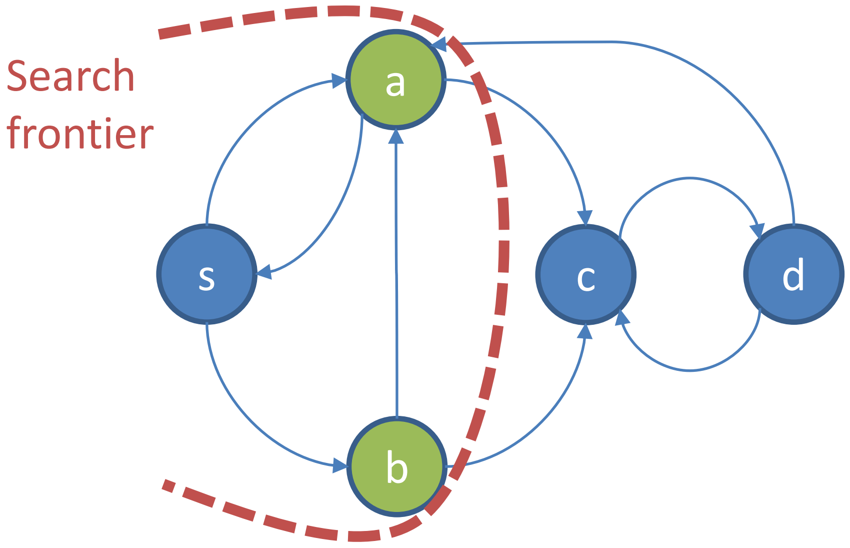

Search frontier
a
s
c
d
b


Search
frontier
a
s
c
d
b


Search
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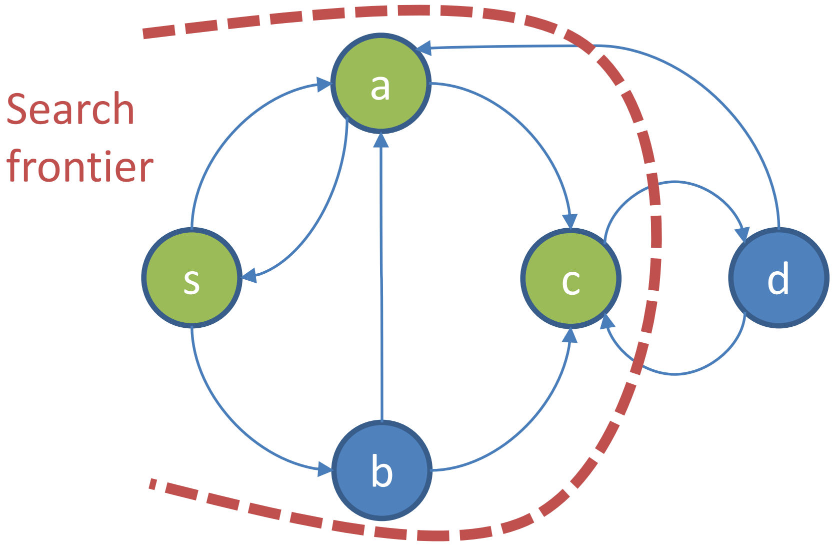
Search
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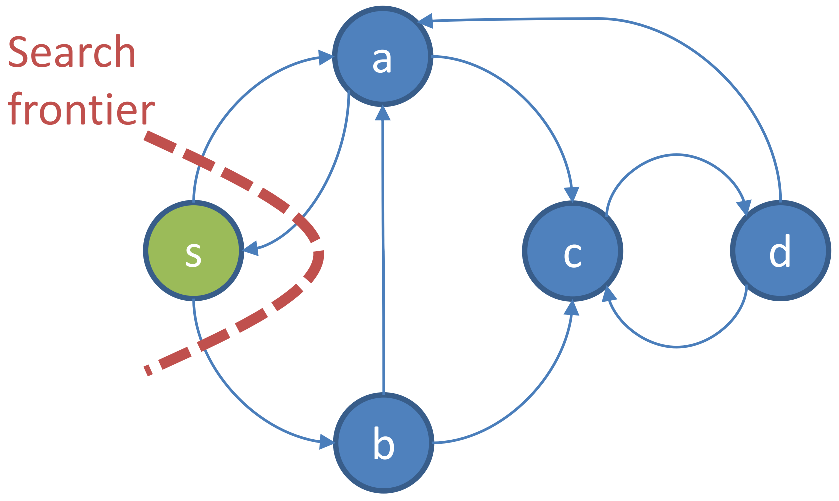
Search frontier
a
s
c
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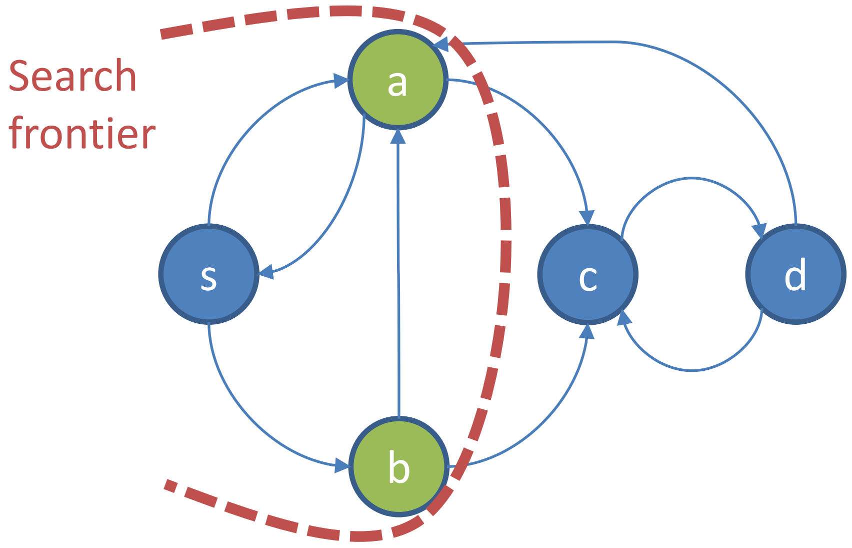
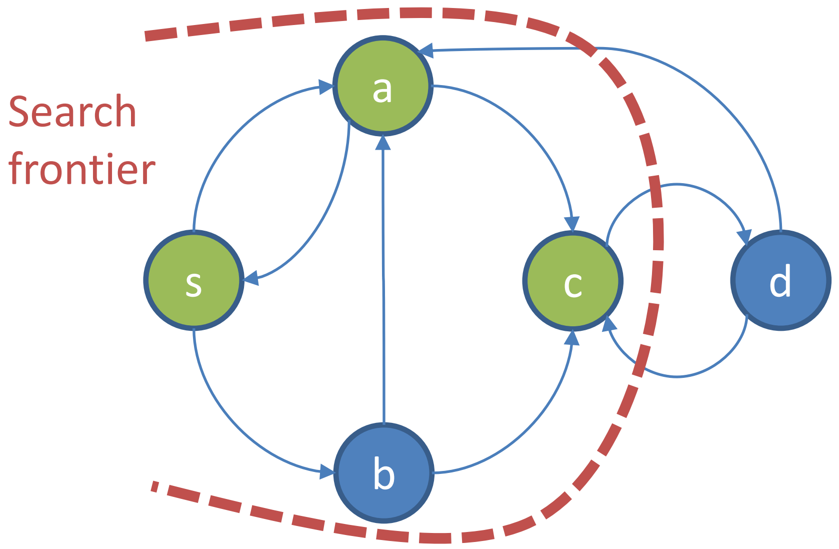
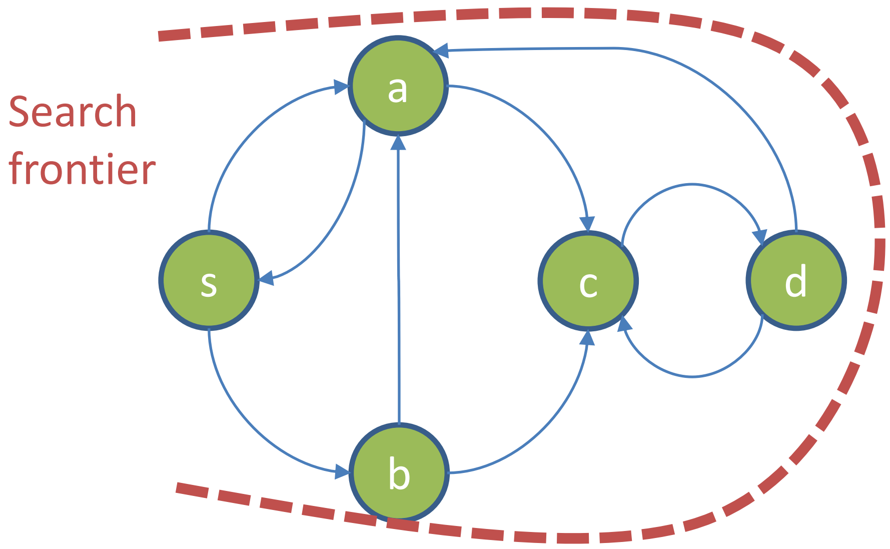
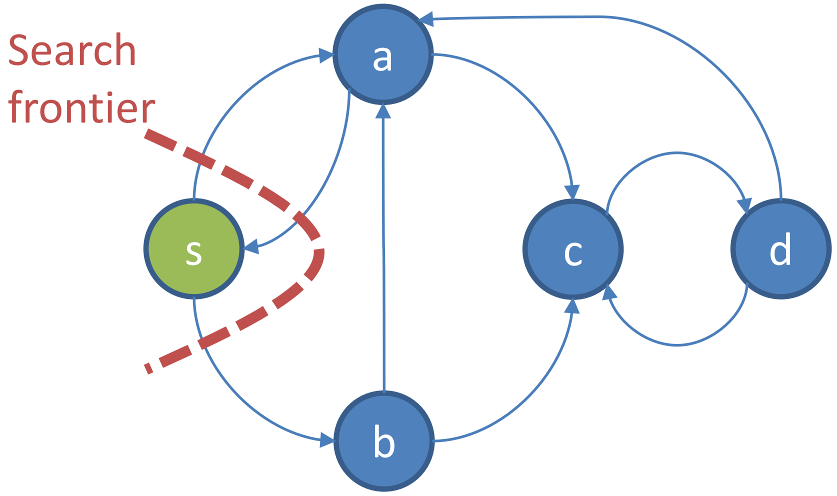

Search frontier
a
s
c
d
b


Search frontier
a
s
c
d
b

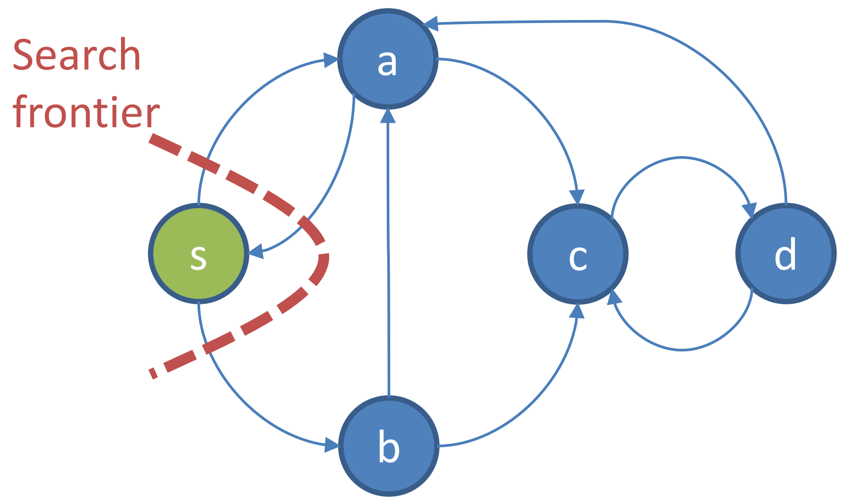

Search frontier
a
s
c
d
b

Parallel BFS

- We work with a graph represented in adjacency-list format. Parallel BFS relies on the following observation: If vertex v was reached in iteration i, then in iteration (i+1) the algorithm must explore all adjacent vertices, i.e., all vertices in v's adjacency list.
  - Each vertex can be processed independently. In the beginning, only source vertex s is "reached," but in later iterations there could be many reached vertices, enabling effective parallelization.
- To turn this idea into an algorithm, let active(i) denote the set of vertices reached in iteration i. The next iteration replaces each active vertex by the vertices in its adjacency list as follows. Each task receives a partition of active(i). For each vertex v in that partition, and for each vertex w in v's adjacency list, emit w. Then active(i+1) is the union of the emitted vertices. This algorithm has two problems:
  - A vertex may be emitted multiple times, requiring shuffling to eliminate duplicates.
  - How can the task access an adjacency list? For large graphs, it is infeasible to copy all adjacency lists (i.e., the entire graph) to all tasks. If a task receives only some adjacency lists, then it can only process those vertices!
- Let us return to our example to see what happens. Assume there are two tasks.



Initially, only source node s is active. The driver informs all tasks in iteration 1 about this.



Initially, only source node s is active. The driver informs all tasks in iteration 1 about this.

Task 0:

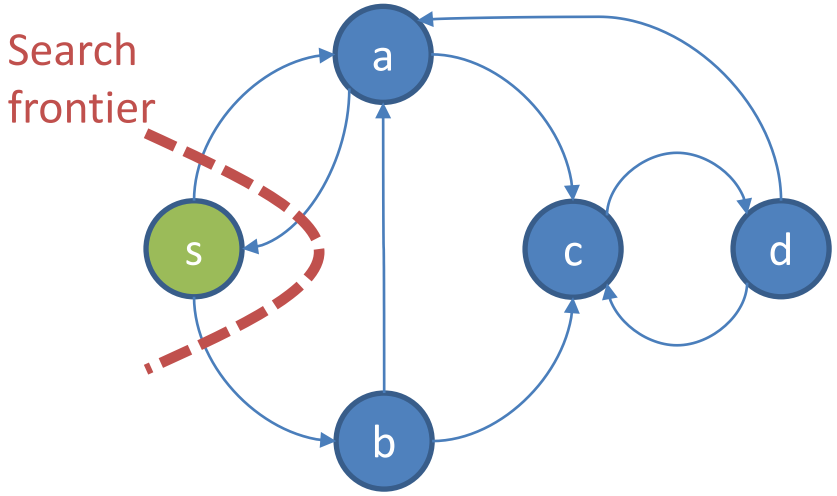
active: **s**
Adjacency lists:
s: a, b
b: a, c
d: a, c

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```

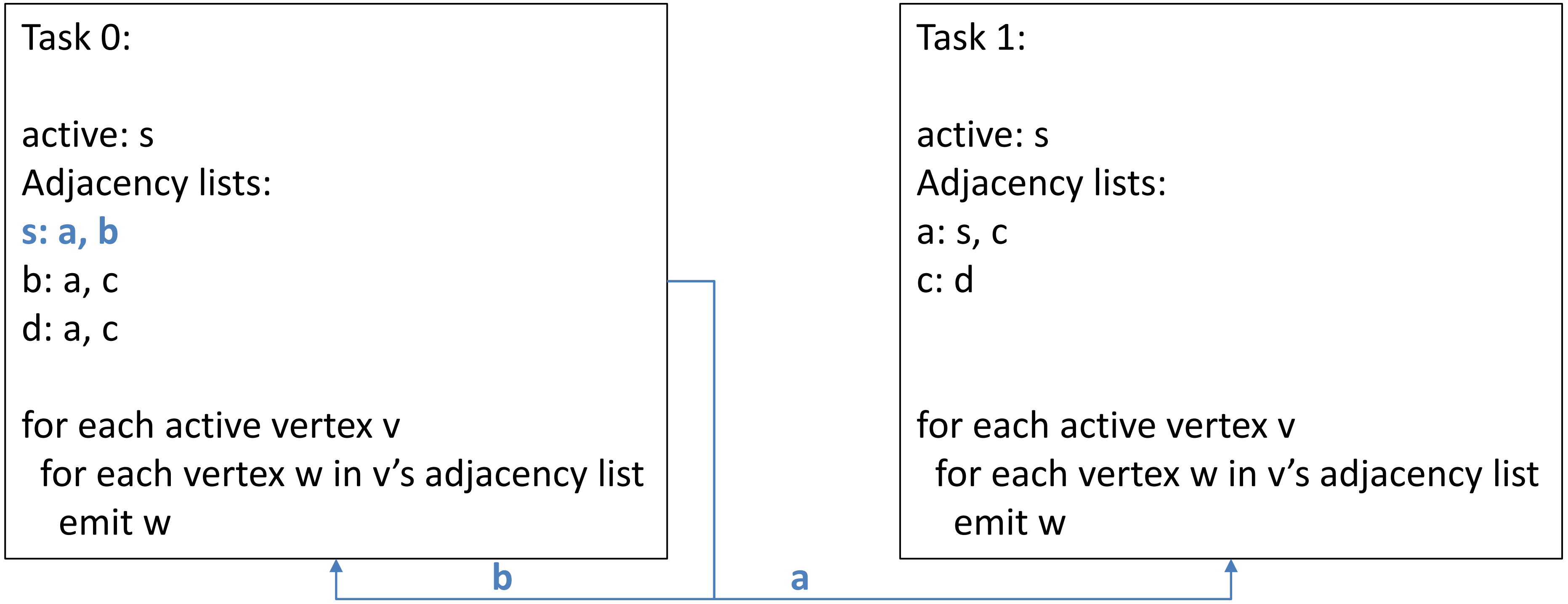
Task 1:

active: **s**
Adjacency lists:
a: s, c
c: d

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```



In iteration 1, all nodes in the adjacency list of s are emitted. This is done by task 0. Task 1 has no active node and hence does not emit anything. The output is sent to the task that owns the corresponding adjacency list.



Task 0:

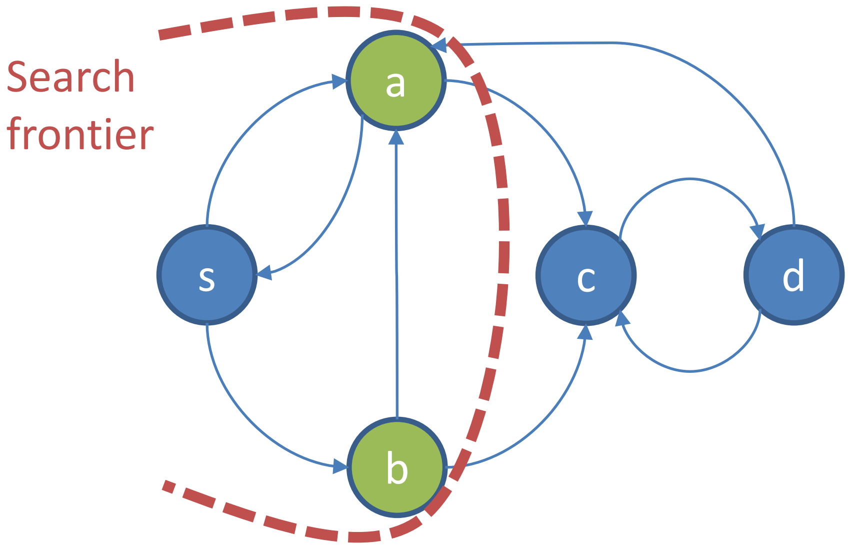
active: s
Adjacency lists:
s: a, b
b: a, c
d: a, c

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```

Task 1:

active: s
Adjacency lists:
a: s, c
c: d

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```



Now vertices a and b are active. Notice that task 0 only knows about “its” vertex b, while task 1 only knows about a.

Task 0:

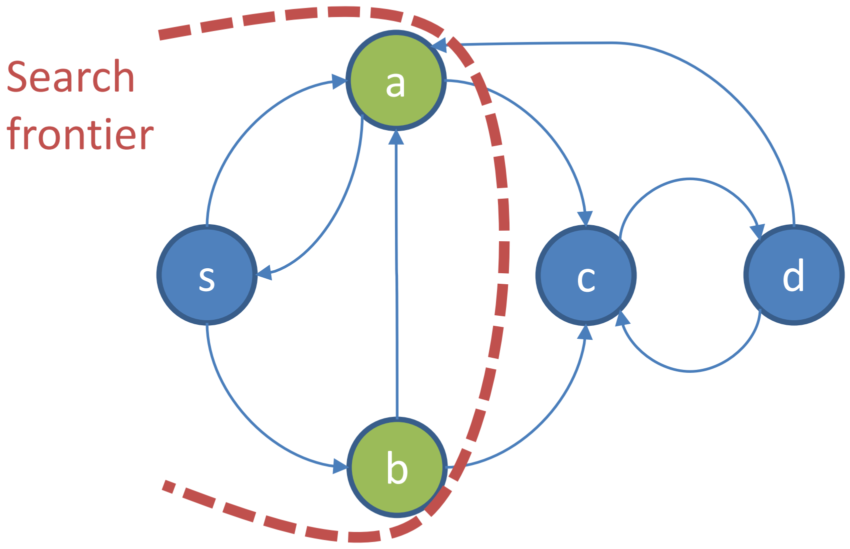
active: **b**
Adjacency lists:
s: a, b
b: a, c
d: a, c

```
for each active vertex v
  for each vertex w in v’s adjacency list
    emit w
```

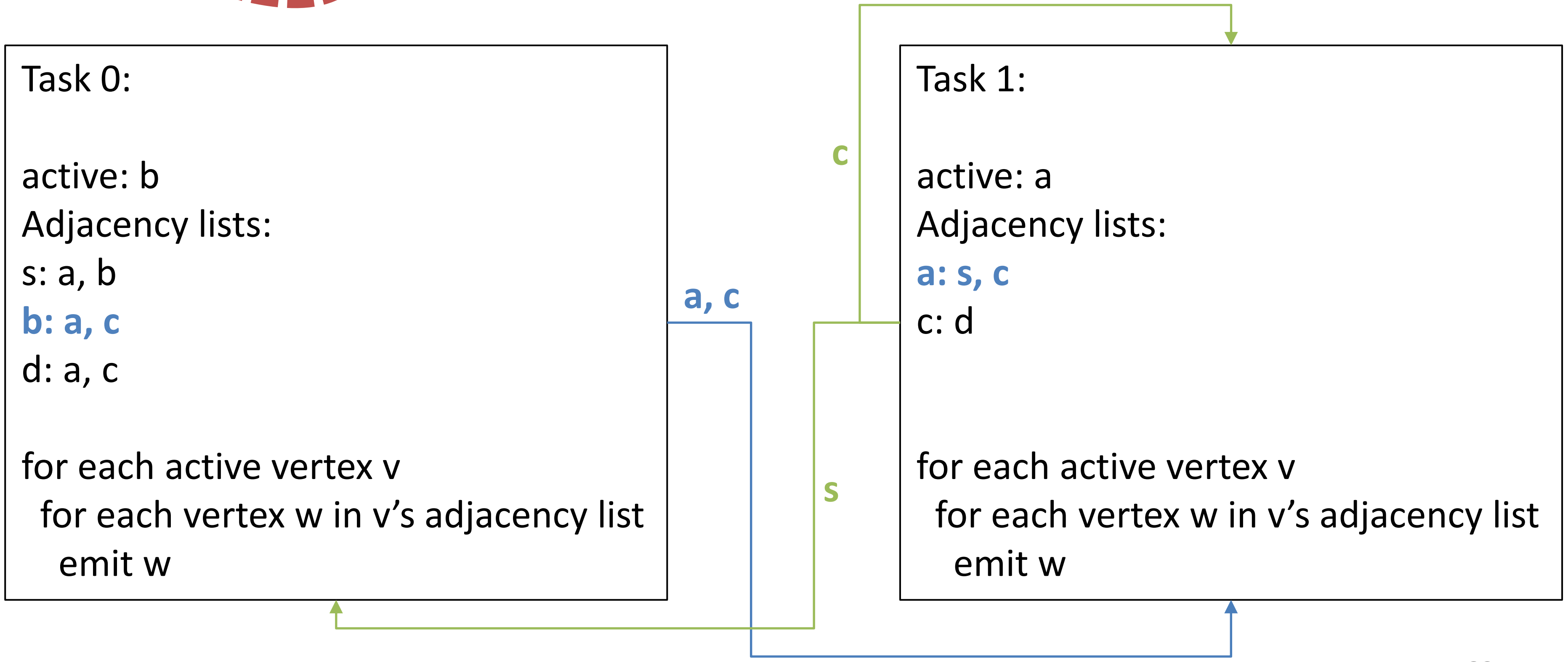
Task 1:

active: **a**
Adjacency lists:
a: s, c
c: d

```
for each active vertex v
  for each vertex w in v’s adjacency list
    emit w
```



In iteration 2, all nodes in the adjacency lists of a and b are emitted. Since a and b are "owned" by different tasks, this work is done in parallel.
Note that c is received twice by task 1. It is straightforward to deal with such duplicates locally.



Task 0:

active: b
Adjacency lists:
s: a, b
b: a, c
d: a, c

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```

Task 1:

active: a
Adjacency lists:
a: s, c
c: d

```
for each active vertex v
  for each vertex w in v's adjacency list
    emit w
```

BFS in MapReduce

- The communication of the emitted active vertices requires shuffling. In MapReduce, this means that each iteration is a full MapReduce job.
- The parallel algorithm requires a partition of the adjacency list to be kept in the memory of a task across iterations. This is *not possible* in MapReduce. Instead, each Map phase must read the adjacency lists again in every iteration.
  - Hence both, newly active vertices and adjacency list data must be read, shuffled, and emitted in every iteration.

BFS in MapReduce

```
// Map processes a vertex object,
// which contains an ID, adjacency list,
// and active status.
map(vertex N) {
  // Pass along the graph structure
  emit(N.id, N)

  // If N is active, mark each vertex in
  // its adjacency list as active
  if (N.isActive)
    for all id m in N.adjacencyList do
      emit(m, "active")
}
```

```
// Reduce receives the vertex object for M and
// between zero or more "active" flags.
reduce(id m, [o1, o2,…]) {
  isReached = false
  M = NULL

  for all o in [o1,o2,…] do
    if isVertexObject(o) then
      // The vertex object was found: recover graph structure
      M = o
    else
      // An active flag was found: m should be active
      isReached = true

  // Update active status of vertex m
  M.setActive( isReached )
  emit(M)
}
```

The driver program repeatedly calls this MapReduce job, each time passing the previous job's output directory as the input to the next.

Avoiding Vertex Revisits

- The BFS algorithm will revisit vertices that are reachable through paths of different lengths from the start vertex. What if we only want to activate newly-reached vertices?
  - We need another flag `wasReached` to keep track of vertices that had been encountered in an earlier iteration.
  - A vertex becomes active only if it is newly reached.

Avoiding Vertex Revisits: Map

```
// Map processes a vertex object, which contains vertex id,
// adjacency list, wasReached flag, and active flag.
// Initially, only the start vertex satisfies wasReached and active.
map(vertex N) {
  // Pass along the graph structure only for vertices not reached yet
  if (not N.wasReached) then
    emit(N.id, N)

  // If N is active, explore its neighbors
  if (N.active) {
    N.active = false

    // Set all neighbors to "reached" status
    for all m in N.adjacencyList do
      emit(m, true)    // true indicates reached status for outlink destination
}
```

Avoiding Vertex Revisits: Reduce

```
// Reduce receives the node object for node M and
// possibly “true” flags if the node was reached from an active node
reduce(id m, [d1,d2,…]) {
 reached = false; M = NULL

 for all d in [d1,d2,…] do
  if isNode(d) then // The vertex object was found: recover graph structure
   M = d
  else // A “true” was found, indicating M was reached from an active vertex
   isReached = true

 // Skip if no vertex object was encountered, i.e., M had been reached earlier
 if (M is not NULL) {  // This guarantees that M.wasReached = false
  // Update reached and active status of M if newly reached
  if (isReached) then {
   M.wasReached = true; M.active = true
  }
  emit(vertex M)
}
```

Improving the MapReduce Algorithm

- Can we avoid the expensive cycle of reading the graph in the Map phase, sending it from Mappers to Reducers, and then writing it in the Reduce phase, which repeats in every iteration? Distributed-file-system accesses are expensive, and the network is a precious shared resource.
- Unfortunately, MapReduce does not support "pinning" data and the corresponding Map and Reduce tasks to a certain worker machine across different MapReduce jobs.
- One could attempt to use the file cache for this purpose. However, it is very limited in the sense that it makes the same data available for each Mapper or Reducer but does not support managing different partitions on different machines.
- In general, MapReduce lacks mechanisms to exploit the repetitive structure of iterative computations. Spark's RDD abstraction addresses exactly this issue.

BFS in Spark

- We discuss the Spark Scala pseudo-code for BFS with RDDs. The version with DataSet is left as a voluntary challenge.
- The Spark program separates static data (the graph structure) from evolving data (the active vertex set).
  - This leverages cached RDD partitions and avoids the shuffling of the adjacency list data we saw in the MapReduce implementation.
  - On the downside, the new active-status information must be joined with the graph data, so that tuples of type (vertexID, adjacencyList) and (vertexID, activeStatus) are joined on vertexID. This join normally requires shuffling, but careful co-partitioning avoids shuffling.

Spark Implementation

```
// Assume the input file contains in each line a vertex ID and its adjacency list.
// Function getAdjListPairData returns a pair of vertex ID and its adjacency list, creating a pair RDD.
graph = sc.textFile(…).map( line => getAdjListPairData(line) )

// Add some code here to make sure that graph has a Partitioner. This is needed for avoiding shuffling
// in the join below.
[…]

// Ask Spark to try and keep this pair RDD around in memory for efficient re-use
graph.persist()

// The active vertex set initially only contains source vertex s, which needs to be passed through the
// context. Create a pair RDD for s. Value 1 is a dummy value.
activeVertices = sc.parallelize( (“s”, 1) )

// Function extractVerticesAsPair returns each vertex id m in the adjacency list as (m, 1),
// where 1 is a dummy value.
for (iterationCount <- 1 to k) {
  activeVertices = graph.join( activeVertices )
            .flatMap( (id, adjList, dummy) => extractVerticesAsPair(adjList) )
            .reduceByKey( (x, y) => x )  // Remove duplicate vertex occurrences
}
```

Real Spark Code

- For an example of BFS, look at the transitive closure program, here the version from the Spark 2.4.0 distribution
  - http://khoury.northeastern.edu/home/mirek/code/SparkTC.scala
- It first generates a random graph and converts it to a pair RDD.
- Note the use of join to extend each path.

BFS in a DBMS

- Assume the graph is stored in table Graph(id1, id2) and the active vertices in table Active(id), where id1, id2, and id are all vertex IDs. Then we can express the computation of an iteration as shown below.
  - This query expresses a left semi-join on Graph.

```
newActive =

          SELECT DISTINCT id2
          FROM Graph AS G
          WHERE G.id1 IN
                    (SELECT * FROM Active)
```

Next, we explore two important graph algorithms: single-source shortest path (SSSP) and PageRank (PR), starting with SSSP.

The parallel versions of both are built on BFS.

Single-Source Shortest Path

- Consider the well-known problem of finding the shortest path from a source vertex s to all other vertices in a graph. The length of a path is equivalent to the total weight of all edges belonging to the path.
- Let all edges in the graph have *non-negative weights*.
- We will first discuss a classic sequential algorithm called Dijkstra’s algorithm, which finds the solution very efficiently. Then we will explore how to solve the problem in parallel.

Dijkstra’s Algorithm

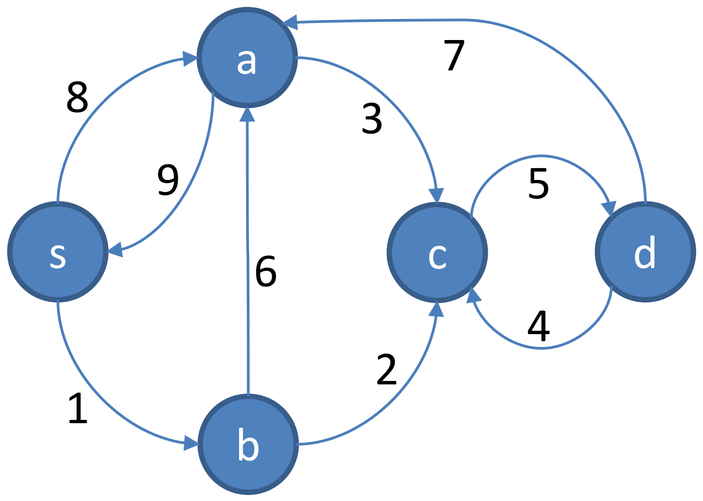
1. Let d[v] denote the distance of vertex v from source s. Initialize d[v] by setting d[s]=0 and d[v]=∞ for all v≠s.
2. Insert all graph vertices into a priority queue sorted by distance. (Initially s will be first, while all other vertices appear in some arbitrary order.)
3. Repeat until the queue is empty
    1. Remove the first vertex u from the queue. Output (u, d[u]). (The shortest path to u was found.)
    2. For each vertex v in u’s adjacency list do
        1. If v is in the queue and d[v] > d[u]+weight(u,v), then set d[v] to d[u]+weight(u,v). (A shorter path to v was found through u.)

Dijkstra’s Algorithm Example

Output:

Priority queue: (s,0), (d,∞), (a,∞), (b,∞), (c,∞)

Vertex being processed:



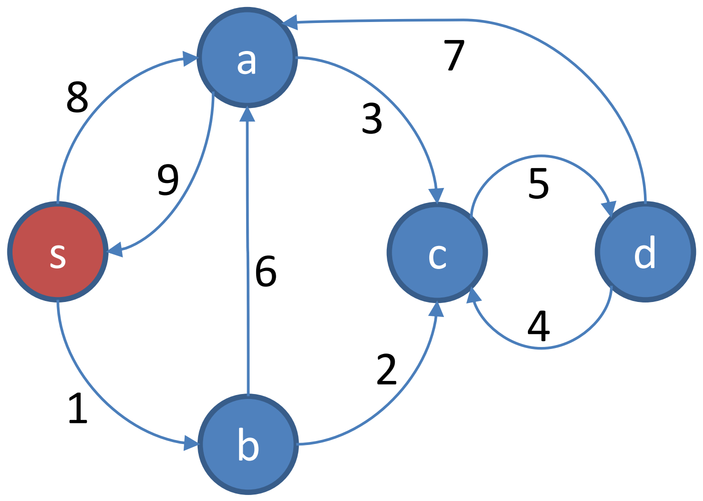
Initially all vertices and their current distance values are in the priority queue.

Dijkstra’s Algorithm Example

Output: (s,0)

Priority queue: ~~(s,0),~~ (d,∞), (a,∞), (b,∞), (c,∞)

Vertex being processed: s: (a,8), (b,1)



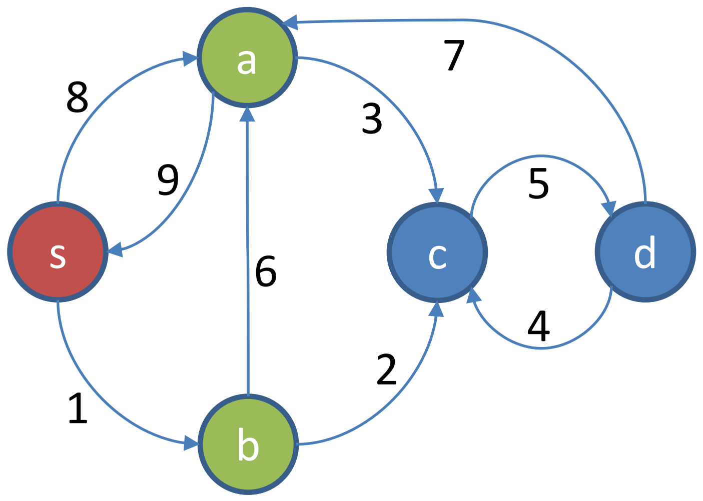
The first vertex, s, is removed and output.

Dijkstra’s Algorithm Example

Output: (s,0)

Priority queue: (b,1), (a,8), (d,∞), (c,∞)

Vertex being processed: s: (a,8), (b,1)



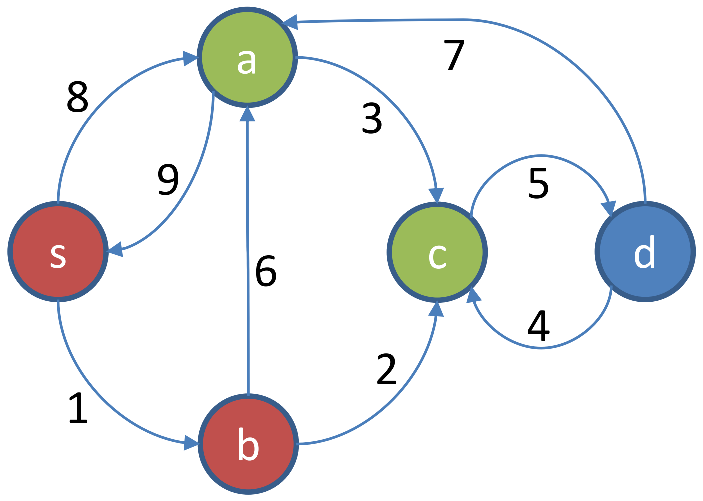
For all vertices in the adjacency list of the removed vertex s, the distances are updated. For example, since d[s]=0 and entry (a,8) appears in the adjacency list, there exists a path to vertex a of length 0+8 = 8.

Dijkstra's Algorithm Example

Output: (s,0), (b,1)

Priority queue: (c,3), (a,7), (d,∞)

Vertex being processed: b: (a,6), (c,2)



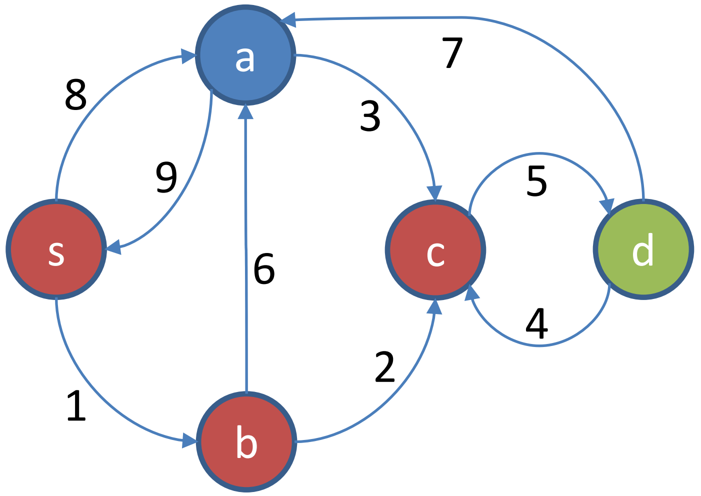
Now vertex b is removed and output, followed by updating of distances of vertices a and c.

Dijkstra’s Algorithm Example

Output: (s,0), (b,1), (c,3)

Priority queue: (a,7), (d,8)

Vertex being processed: c: (d,5)



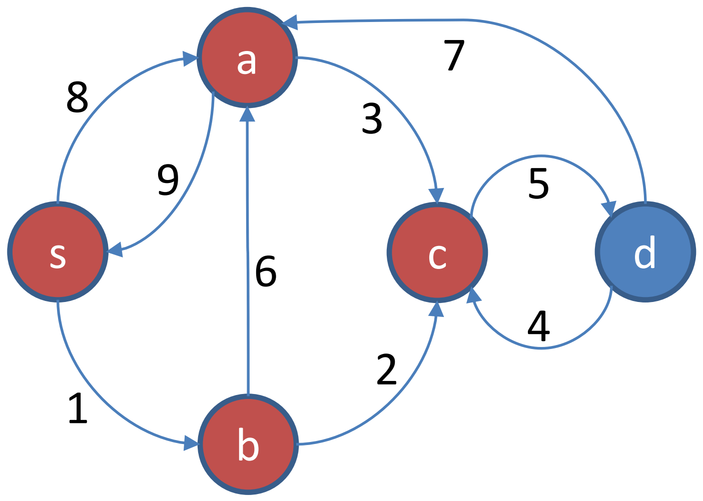
Now vertex c is removed and output, followed by updating of the distance of vertex d.

Dijkstra’s Algorithm Example

Output: (s,0), (b,1), (c,3), (a,7)

Priority queue: (d,8)

Vertex being processed: a: (c,3), (s,9)



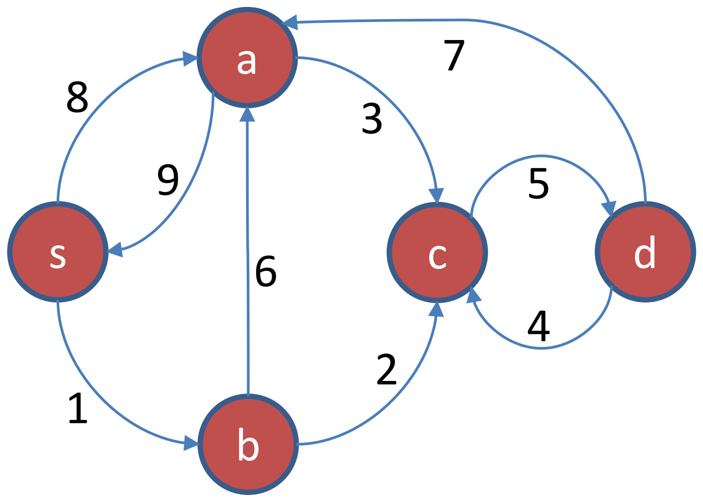
Now vertex a is removed and output. Since vertices c and s in a’s adjacency list are not in the queue, no distance updates are performed.

Dijkstra’s Algorithm Example

Output: (s,0), (b,1), (c,3), (a,7), (d,8)

Priority queue:

Vertex being processed: d: (c,4), (a,7)



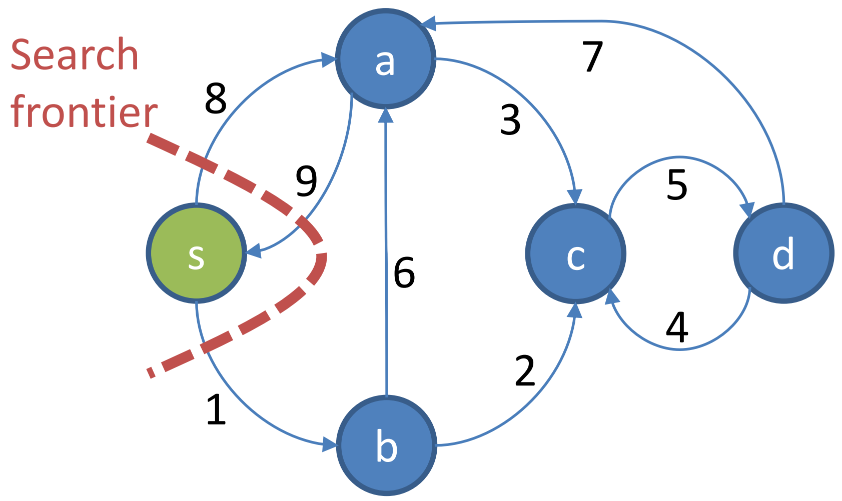
Finally vertex d is output and the algorithm terminates.

Parallel Single-Source Shortest Path

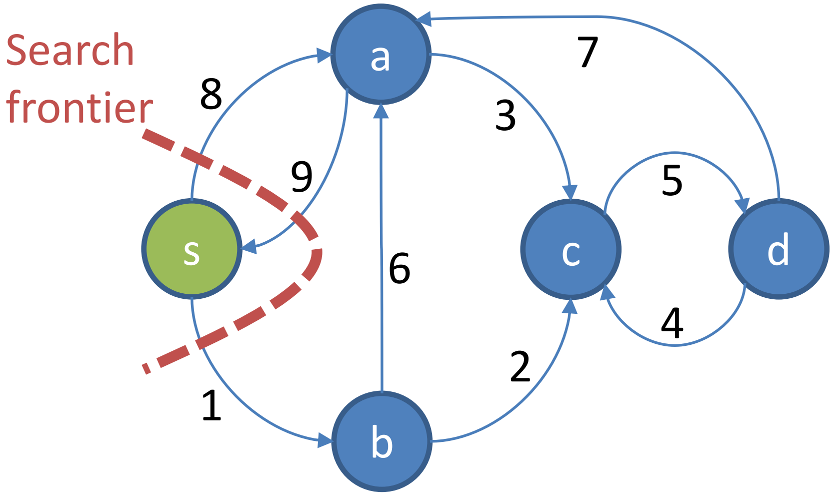
- Dijkstra’s algorithm is elegant and very efficient for sequential execution, but difficult to adapt for parallel execution: vertices are removed from the priority queue one-by-one. One cannot remove multiple vertices at once for parallel processing, without risking incorrect results.
  - Recall the example where after processing vertex s, entries (b,1) and (a,8) were the first entries in the queue. Removing both and processing them in parallel would not have worked, because the shortest path to a is going through b.
- While removing multiple vertices at once jeopardizes correctness in *any* parallel environment, the use of a single priority queue represents a particular challenge for MapReduce and Spark. It is not clear how to best implement such a shared data structure in a system without an efficiently implemented shared-memory abstraction.

Shortest Path using BFS

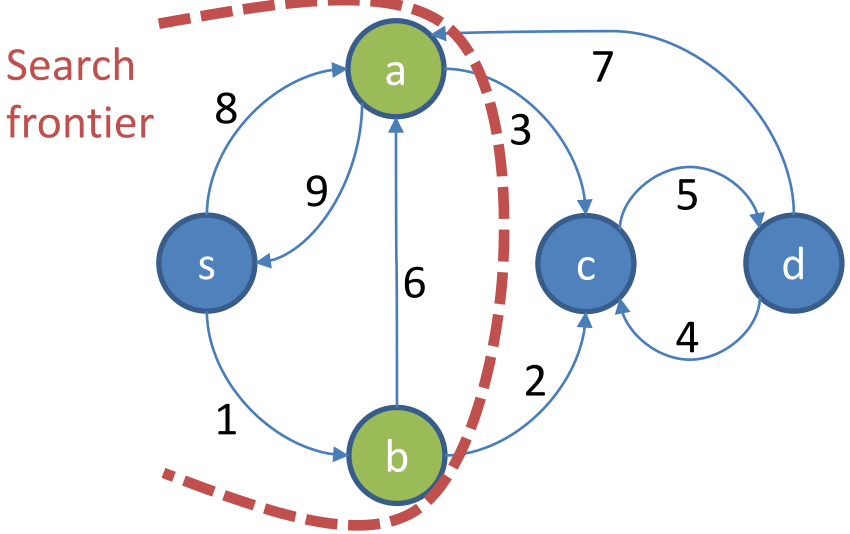
- Even without the priority queue, which is the key component of Dijkstra’s algorithm, the parallel solution can still rely on the following property: If there exists a path of length d[u] to some vertex u, then for each vertex v in u’s adjacency list there exists a path of length d[u]+weight(u,v).
- We design an algorithm that exploits this property by systematically exploring the graph using BFS:
  - In the first iteration, find all vertices reachable from s in exactly one hop and update their distances.
  - In the second iteration, find all vertices reachable from s in exactly two hops and update their distances. And so on.
  - Continue this process until the shortest path to each vertex was found.
- We illustrate the algorithm next for the same example graph.



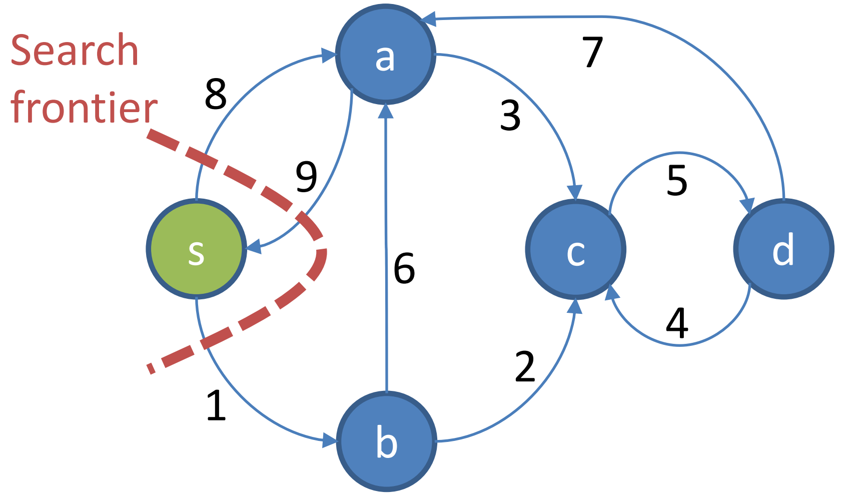
Initial state: (s,0), (a,∞), (b,∞), (c,∞), (d,∞)



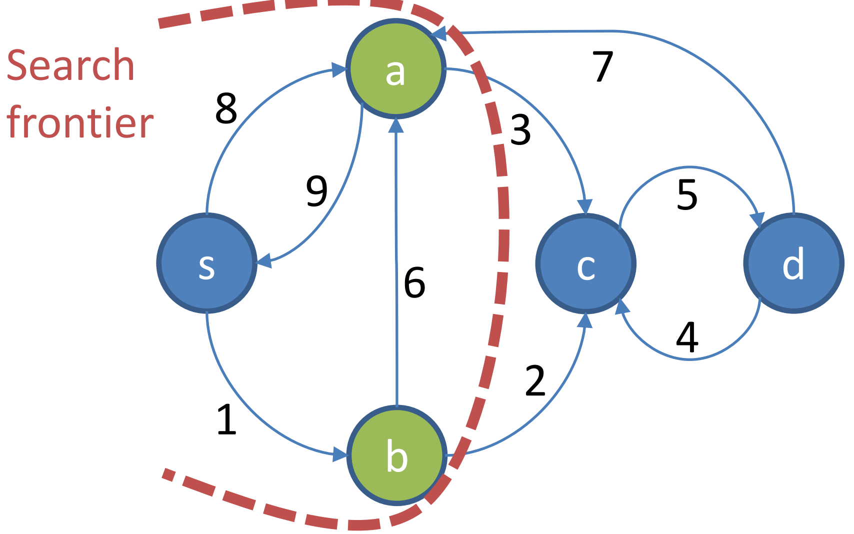
Initial state: (s,0), (a,∞), (b,∞), (c,∞), (d,∞)



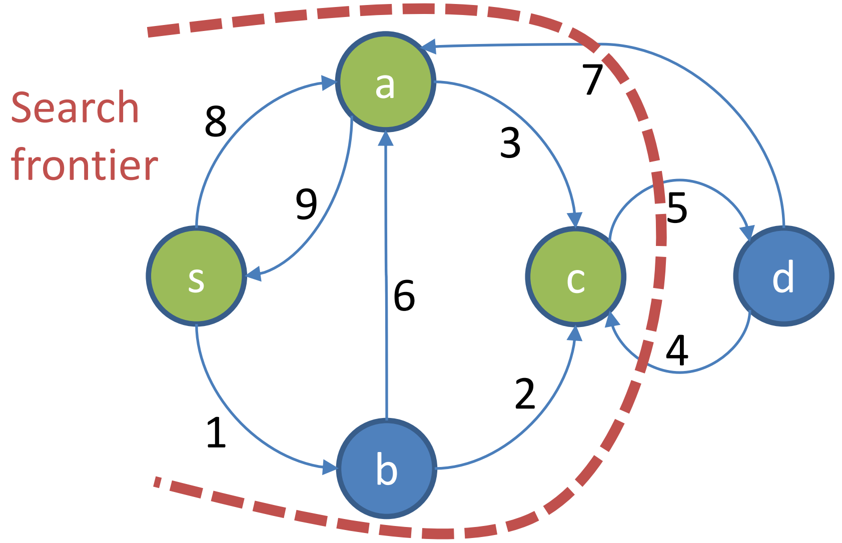
Iteration 1: (s,0), (a,8), (b,1), (c,∞), (d,∞)



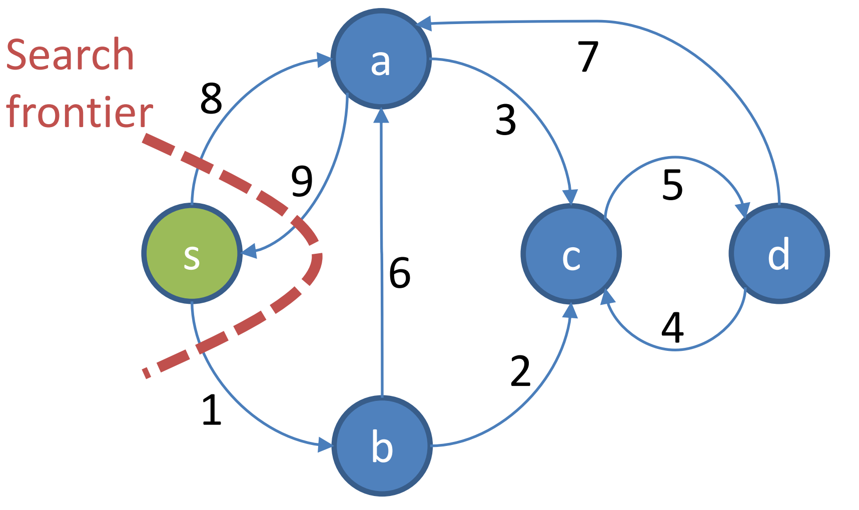
Initial state: (s,0), (a,∞), (b,∞), (c,∞), (d,∞)



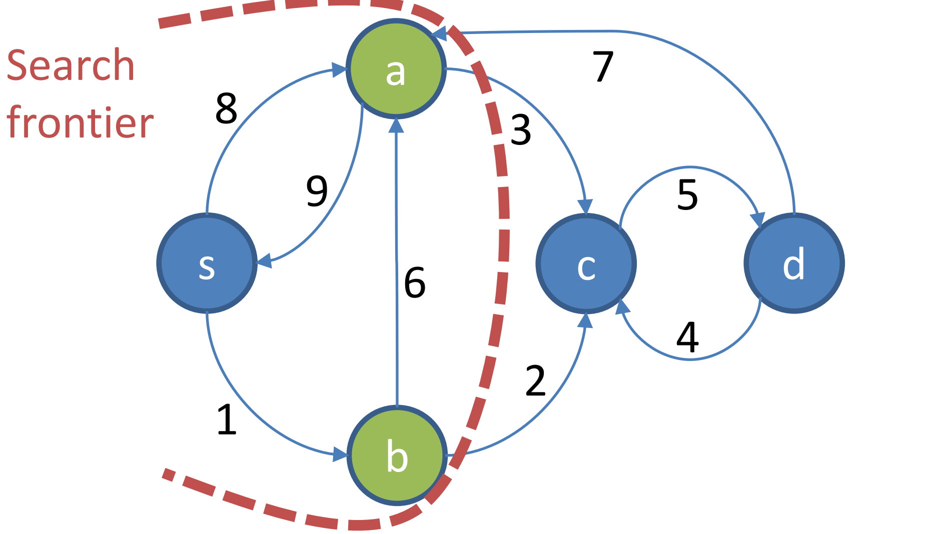
Iteration 1: (s,0), (a,8), (b,1), (c,∞), (d,∞)



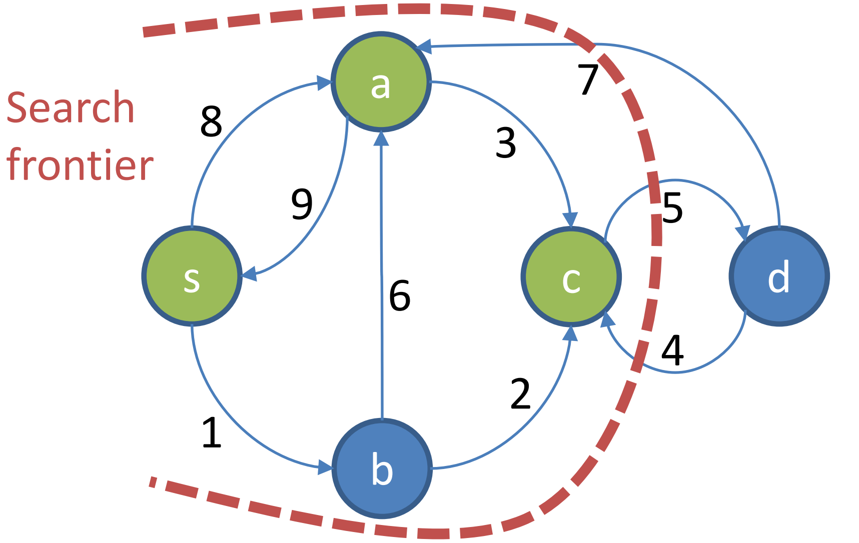
Iteration 2: (s,0), (a,7), (b,1), (c,3), (d,∞)



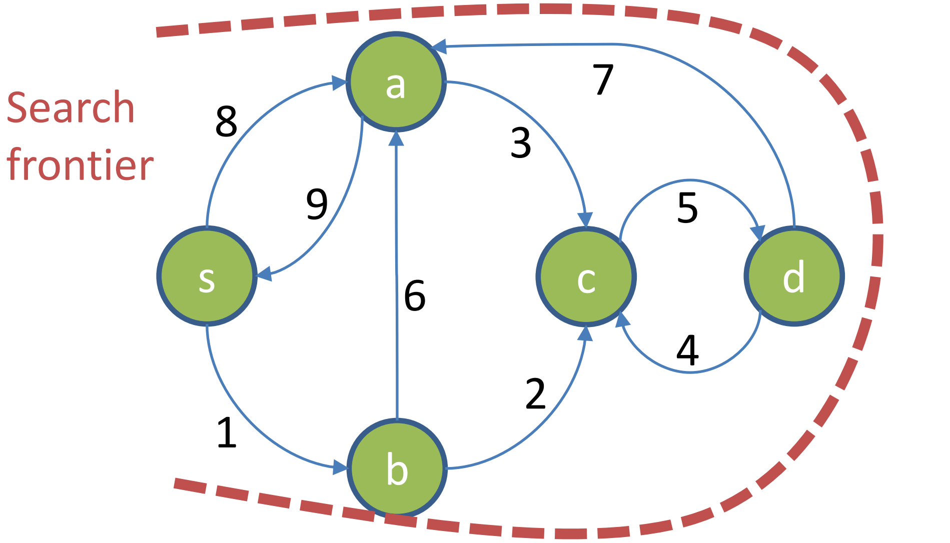
Initial state: (s,0), (a,∞), (b,∞), (c,∞), (d,∞)



Iteration 1: (s,0), (a,8), (b,1), (c,∞), (d,∞)



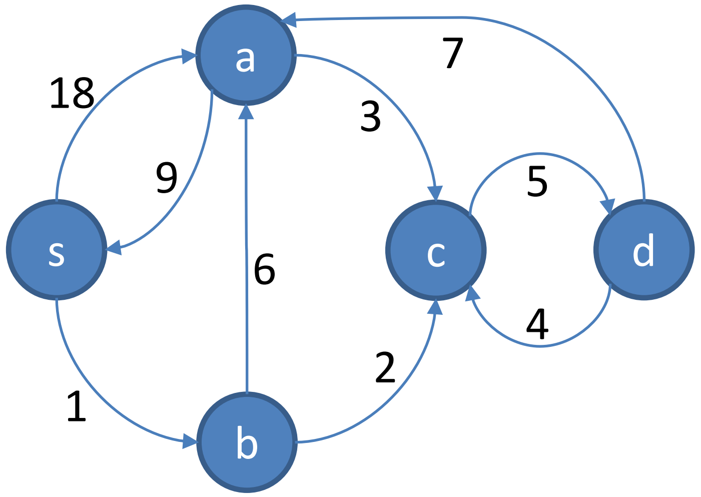
Iteration 2: (s,0), (a,7), (b,1), (c,3), (d,∞)



Iteration 3: (s,0), (a,7), (b,1), (c,3), (d,8)

When to Stop Iterating

- How does the algorithm know when all shortest paths were found?
- If all edges have the same weight, stop iterating as soon as no vertex has distance ∞ any more: Later iterations can only find longer paths. This can be detected using a global counter or accumulator.
  - The number of iterations depends on the graph diameter. In practice, social networks often show the small-world phenomenon, i.e., have a small diameter.
- If edges have different weights, then a "detour" path can have a lower total weight than a more "direct" connection and hence we cannot stop as soon as all vertex distances are finite. Instead, iterations must continue until no shortest distance changes. This can also be detected using a global counter or accumulator.
  - In the worst case, this takes |V|-1 iterations, where |V| is the number of graph vertices.
- In a graph with cycles of negative weight the algorithm never terminates: each traversal of the cycle reduces the distances of all its vertices indefinitely.



This is an example for a graph where a "detour" path consisting of four edges is shorter than the direct path. Path (s, a) has length 18, while path (s, b, c, d, a) has length 15. Detecting this shortest path requires 4 iterations.

Single-Source Shortest Path in MapReduce

- The algorithm is virtually identical to BFS, but also keeps track of the shortest distance found for each vertex so far:
  - Map processes a single vertex u, emitting d[u]+weight(u,v) for each vertex v in u's adjacency list.
  - Reduce collects the newly computed distances for all inlinks of vertex v and determines if any of them is shorter than its currently known shortest distance.
- The driver program repeatedly calls the MapReduce program as many times as necessary, exploring ever longer paths.

MapReduce Code for a Single Iteration

```
// The vertex object now also contains the
// current min distance and edges in the
// adjacency list have a weight
map(vertex N) {
  // Pass along the graph structure
  emit(N.id, N)

  // Compute the distance for each outlink
  for all m in N.adjacencyList do
    emit( m, N.distance + weight(n,m) )
}
```

```
// Reduce receives the vertex object for vertex m and
// the newly computed distances for m's inlinks
reduce(id m, [d1, d2,…]) {
  dMin = ∞
  M = NULL

  for all d in [d1, d2,…] do
    if isVertex(d) then
      // The vertex object was found: recover graph structure
      M = d
    else
      // A distance value for an inlink was found: keep track
      // of the minimum.
      if d < dMin then dMin = d

  // Update distance of vertex m if necessary.
  if dMin < M.distance then M.distance = dMin
  emit( m, M )
}
```

MapReduce Algorithm Analysis

- Each iteration of the algorithm performs a large amount of work:
  - The entire graph is read from the distributed file system, transferred from Mappers to Reducers, and then, with updated distance values, written to the distributed file system.
  - For every vertex u, no matter if it potentially lies on a shorter path to another vertex v or not, distance d[u]+weight(u,v) is computed and sent to the Reducers.
  - For every vertex v, no matter if its shortest path was already found in previous iterations or not, the Reduce function is executed to re-compute the shortest distance.
- This brute-force approach performs many irrelevant computations:
  - In early iterations, Map computes distances for vertices that have not yet been reached, therefore still have infinity distance.
  - In later iterations, the program keeps re-computing paths for vertices whose shortest path was already found.
- Contrast this with the elegance of Dijkstra’s algorithm, which avoids this irrelevant computation, but needed the priority queue in order to achieve this.

Improving the MapReduce Algorithm: Avoiding Useless Work

- Can we avoid processing vertices that do not improve any distances in the iteration? This turns out to be easy based on the following observations:
  - If a vertex u has distance d[u]=∞, then it cannot help reduce the distance for any of the vertices in its adjacency list.
  - Assume vertex u had the same distance d[u]=x in iterations i and (i+1). For any vertex v in its adjacency list, the Map function call for u would emit the same value x+weight(u,v) in both iterations. Since this value was already included in the Reduce computation in iteration i, it cannot result in improvements in iteration (i+1).
- To exploit these properties, we can do the following:
  - Like BFS, the program distinguishes between “active” and “inactive” vertices. Active vertices are those that could potentially help reduce the distance for another vertex. We define a vertex to be active if and only if its distance value changed in the previous iteration. The only exception to this rule is source vertex s, which is set to “active” before the first iteration. Note that a vertex that was active in one iteration could become inactive in the next, and vice versa.
  - It is easy to prove that a vertex whose distance reached the final value, i.e., the shortest-path distance from s, will remain inactive afterwards. Hence the algorithm can stop iterating as soon as all vertices become inactive.
- The corresponding program is shown next.

Improved MapReduce Code for a Single Iteration

```
// The vertex object now also keeps track
// if the vertex is active.
map( vertex N ) {
  // Pass along the graph structure
  emit(N.id, N)

  // Compute the distance for each outlink
  // of an active vertex
  if N.isActive {
    for all m in N.adjacencyList do
      emit( m, N.distance + weight(n,m) )
  }
}
```

```
// Reduce receives the vertex object for vertex m and
// the newly computed distances for m's inlinks
reduce(id m, [d1, d2,…]) {
  dMin = ∞;  M = NULL

  for all d in [d1, d2,…] do
    if isVertex(d) then
      // The vertex object was found: recover graph structure
      M = d
    else
      // A distance value for an inlink was found: keep track
      // of the minimum.
      if d < dMin then dMin = d

  // Update distance of vertex m if necessary.
  if dMin < M.distance then {
    M.distance = dMin
    // The distance change can affect the distance for
    // nodes in the adjacency list, hence set status to active
    M.setActive(true)
  }
  emit( m, M )
}
```

Single-Source Shortest Path in Spark

- We discuss the Spark Scala pseudo-code with RDDs. The version with DataSet is left as a voluntary challenge.
- The Spark program separates static data (the graph structure) from evolving data (the currently known shortest distance of each vertex).
  – This leverages cached RDD partitions and avoids the shuffling of the adjacency list data we saw in the MapReduce implementation.
  – On the downside, the new distance-status information must be combined with the graph data, so that tuples of type (vertexID, adjacencyList) and (vertexID, distance) are joined on vertexID. This join normally requires shuffling, but careful co-partitioning avoids shuffling.

Spark Implementation

```
// Each line in the input file contains a vertex ID and its adjacency list. Function getAdjListPairData returns
// a pair (vertex ID, adjacency list), creating a pair RDD. An entry in the adjacency list is a pair of destination
// vertex id and edge weight. The source node's adjacency list has a special "source" flag for setting of
// the initial distances.
graph = sc.textFile(…).map( line => getAdjListPairData(line) )

// Add code here to make sure the graph has a Partitioner. This avoids shuffling in the join below.

// Tell Spark to try and keep this pair RDD in memory for efficient re-use
graph.persist()

// Create the initial distances. mapValues ensures that the same Partitioner is used as for the graph RDD.
distances = graph.mapValues( (id, adjList) => hasSourceFlag(adjList) match {
                                               case true => 0
                                               case _ => infinity }

// Function extractVertices returns each vertex id m in n's adjacency list as (m, distance(n)+w),
// where w is the weight of edge (n, m). It also returns n itself as (n, distance(n))
for (iterationCount <- 1 to k) {      // Use Accumulator instead to determine when last iteration is reached
  distances = graph.join( distances )
        .flatMap( (n, adjList, currentDistanceOfN) => extractVertices(adjList, currentDistanceOfN) )
        .reduceByKey( (x, y) => min(x, y) )      // Remember the shortest of the distances found
}
```

Single-Source Shortest Path in a DBMS

- Assume the graph is stored in table Graph(id1, id2, weight) and the currently known shortest distances in table Distances(id, distance), where id1, id2, and id are all vertex IDs. Then we can express the computation of an iteration as shown below.

```
tempDistances =
        SELECT G.id2 AS id, D.distance+G.weight AS distance
        FROM Graph AS G, Distances AS D
        WHERE G.id1 = D.id
        UNION
        SELECT * FROM Distances

newDistances =
        SELECT id, min(distance)
        FROM tempDistances
        GROUP BY id
```

We now explore the famous PageRank (PR) algorithm.

Compare and contrast the parallel version to SSSP.

PageRank

- PageRank was popularized by Google as a measure for evaluating the importance of a Web page. Intuitively it assigns greater importance to pages that are linked from many other important pages.
- PageRank captures the probability of a "random Web surfer" landing on a given page. The random Web surfer can reach a page by jumping to it or by following the link from another page pointing to it.
- PageRank helps identify the most relevant results for a keyword query. For example, consider query "Northeastern University." A person entering these keywords will most likely be looking for the Northeastern.edu homepage. If a search engine only considers traditional measures of importance such as TF/IDF, it might highly rank a spam page containing term "Northeastern" many times. Assuming that Northeastern.edu will be linked from many more important pages than the spam site, taking into account the PageRank value can help boost its rank in the result list.

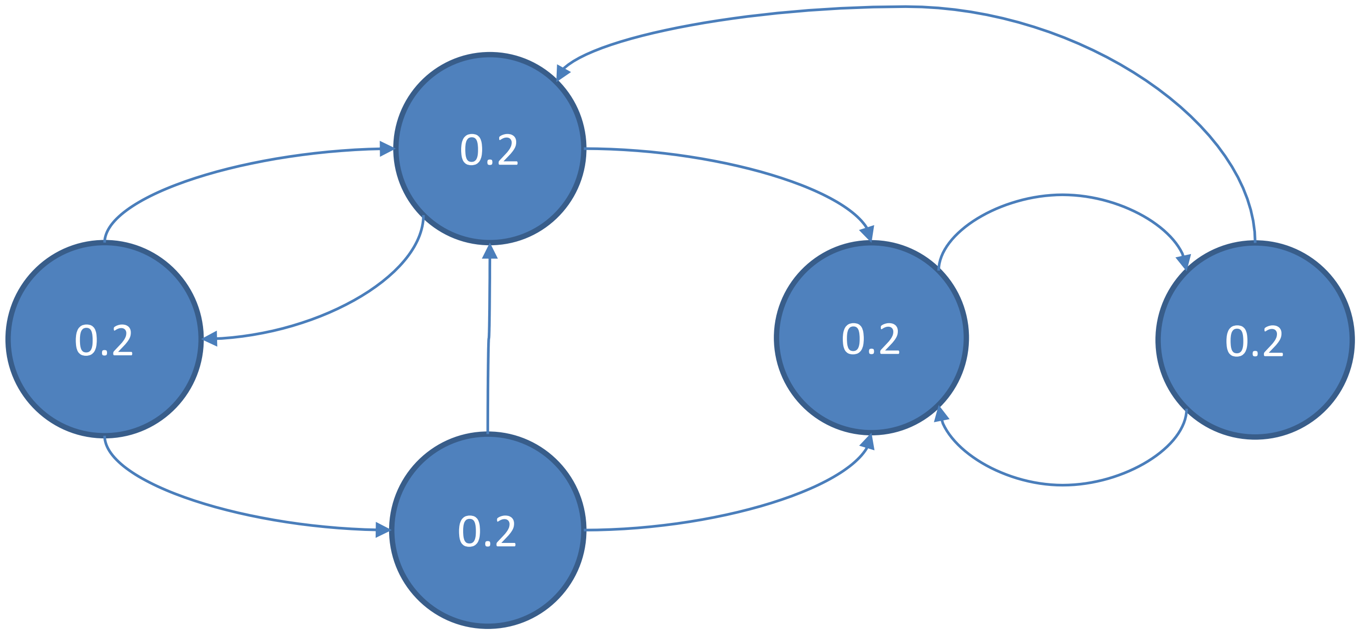
PageRank Definition

- The importance of a Web page is measured by the probability that a random Web surfer will land on it. They can reach a page either by typing a random URL into the browser's address field (aka perform a random jump) or by following a random link from the page they currently visit. The probability of doing the latter is α.
- Formally, $P(n) = (1-\alpha)\frac{1}{|V|} + \alpha \sum_{m \in L(n)} \frac{P(m)}{C(m)}$, where
  - |V| is the number of pages (vertices) in the Web graph considered.
  - α is the probability of the surfer following a link (vs. 1-α for a random jump).
  - L(n) is the set of all pages in the graph linking to n.
  - P(m) is the PageRank of another page m.
  - C(m) is the out-degree of page m, i.e., the number of links on that page.
- The 2 main terms in the formula correspond to the 2 ways of reaching n:
  - The random surfer reaches it via a random jump if they (1) decide to make a random jump (probability 1-α) and (2) choose exactly this 1 out of |V| possible Web pages.
  - They reach it via a link if they (1) decide to follow a link (probability α), (2) were visiting another page m (which they happen to be currently visiting with probability P(m)) that links to n, and (3) selected the 1 out of C(m) links on m pointing to n.

Iterative PageRank Computation

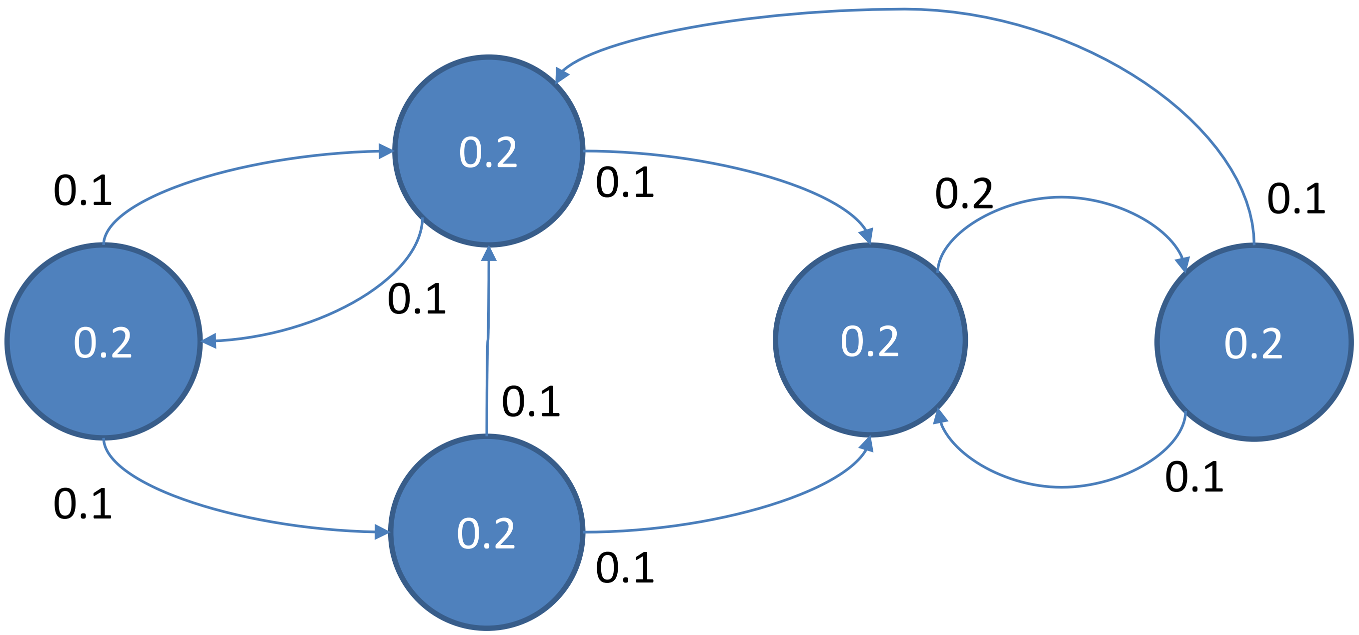
- Notice that the definition of PageRank creates a "chicken-and-egg problem":
  - To compute the PageRank of page n, we need to know the PageRank of all other pages linking to it, which in turn might depend on n's PageRank.
- Fortunately, this recursive definition admits an iterative algorithm for computing all PageRank values in the graph. Starting with some initial values, each iteration computes the new PageRank for all pages. This process continues until a fixpoint is reached, meaning that the value for every page does not change any more.
- Let's look at an example to see PageRank iterations in action.

Initial State



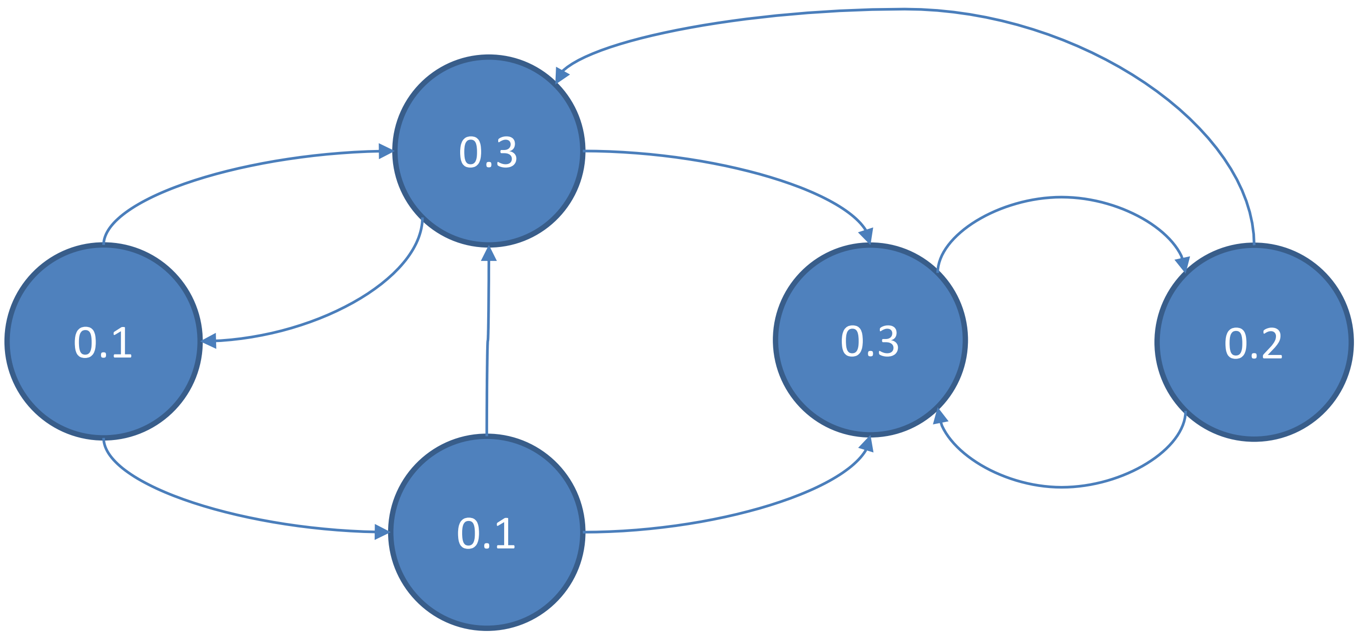
Assume that all PageRank values are initialized to 0.2. For simplicity we set α=1, i.e., the random surfer only follows links.

Iteration 1: PageRank Transfer



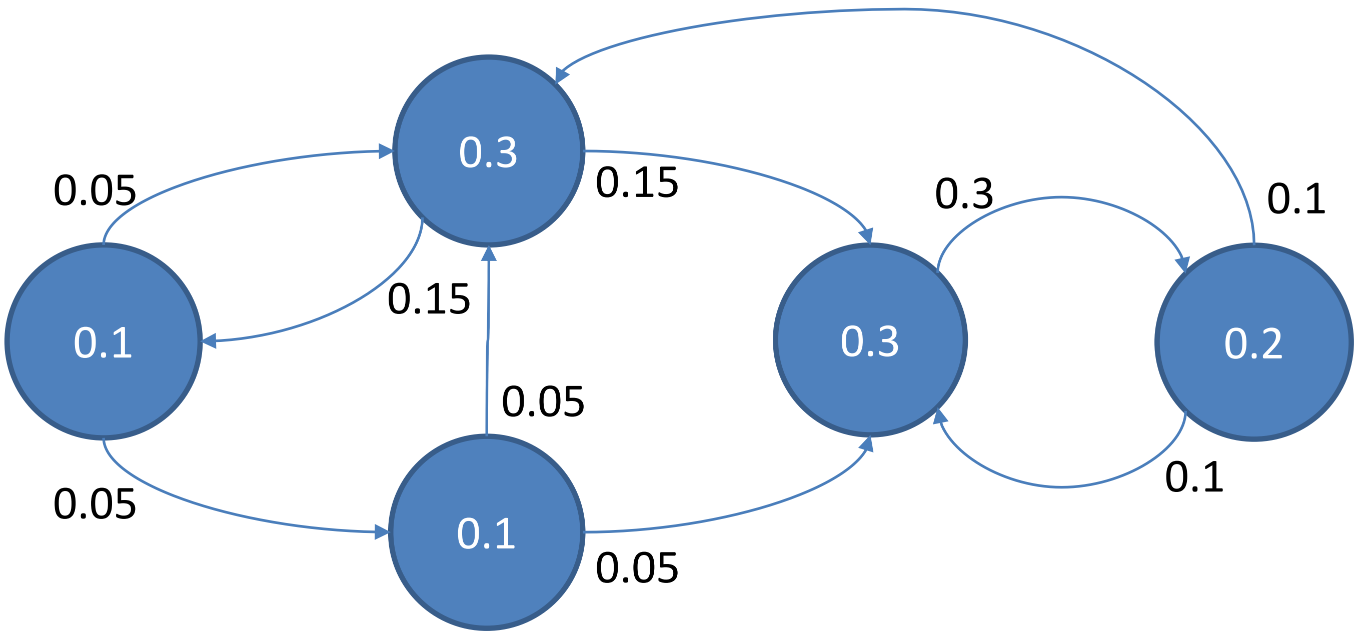
Since α=1, each page passes on its full PageRank value, distributed equally over the outgoing links.

Iteration 1: Updated PageRank



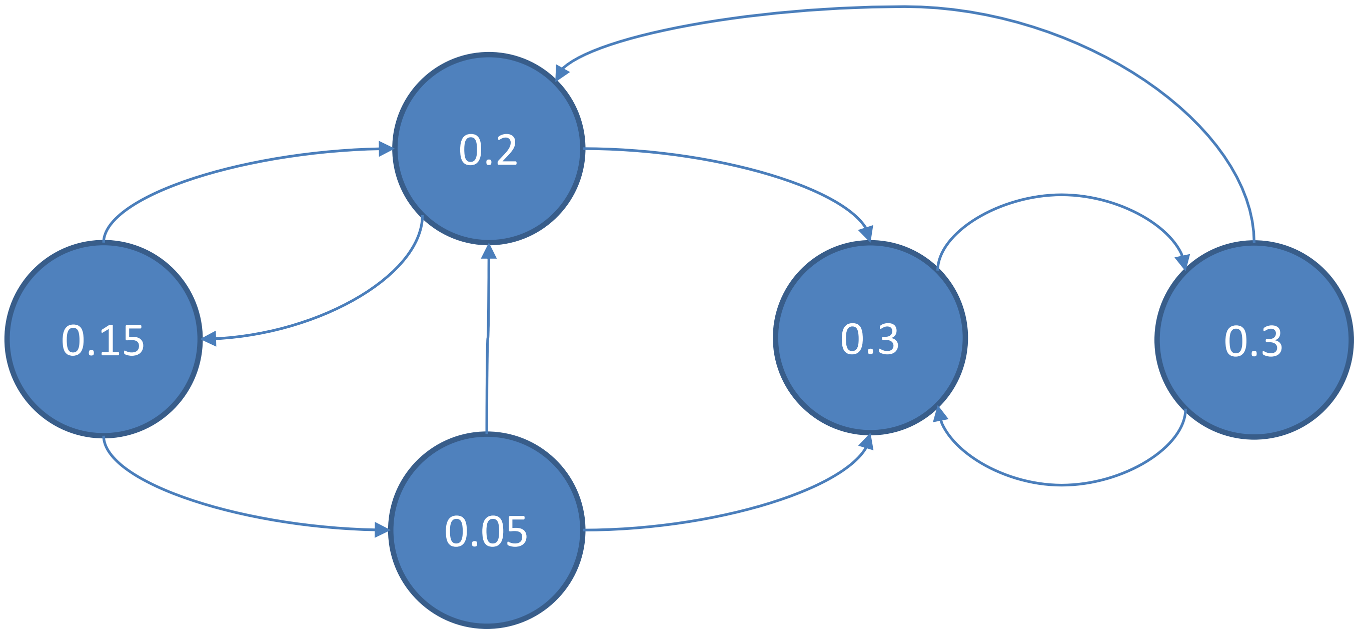
It is clearly visible how some pages receive more weight than others.

Iteration 2: PageRank Transfer



Since α=1, each page again passes on its full PageRank value along the outgoing links.

Iteration 2: Updated PageRank



Already after two iterations, major properties of the algorithm show. First, from iteration to iteration, the PageRank of a page can oscillate between higher and lower values. E.g., the leftmost page changed from 0.2 to 0.1, then to 0.15. Over time, these changes become smaller as the values converge. Second, despite the oscillations, the general tendency is for some pages to accumulate larger values, while others drop.

PageRank using BFS

- Observing the steps during an iteration of PageRank, it becomes clear that there are two phases. In the first phase of the iteration, a page sends out fractions of its current PageRank along its *outgoing* edges. In the second phase, each page sums up the PageRank contributions along all its *incoming* edges.
- Given page n's current value P(n) and adjacency list, one can compute all its outgoing contributions.
- Adding the incoming contributions for page m requires re-shuffling.
  - The term $(1-\alpha)/|V|$ remains constant throughout the computation. Both α and $|V|$ can be shared with all tasks.
- Interestingly, the computation pattern again matches BFS.

PageRank in MapReduce

- The algorithm is virtually identical to BFS, but an iteration must also keep track of the current PageRank value of each page:
  - Map processes a single vertex n, emitting the PageRank of n, divided by n's outdegree, for each vertex m in n's adjacency list.
  - Reduce collects PageRank contributions from all inlinks of vertex m and then applies the formula.
- The driver program repeatedly calls the MapReduce program until (near) convergence, i.e., when all PageRank values stabilize.

MapReduce Code for a Single Iteration

```
// The vertex object now also stores
// the current PageRank value
map( vertex N ) {
  // Pass along the graph structure
  emit(N.id, N)

  // Compute contributions to send
  // along outgoing links
  p = N.pageRank / N.adjacencyList.size()
  for all m in N.adjacencyList do
    emit( m, p )
}
```

```
// Reduce receives the vertex object for vertex m and
// the PageRank contributions for all m's inlinks
reduce(id m, [p1, p2,…]) {
  s = 0
  M = NULL

  for all p in [p1, p2,…] do
    if isVertex(p) then
      // The vertex object was found: recover graph structure
      M = p
    else
      // A PageRank contribution from an inlink was found:
      // add it to the running sum.
      s += p

  M.pageRank = $(1-\alpha)/|V| + \alpha \cdot s$
  emit( m, M )
}
```

MapReduce Algorithm Analysis

- A careful look reveals that this program is structurally almost identical to the one for single-source shortest path. Hence it shares the same weaknesses caused by MapReduce's inability to exploit repetitive structure in iterative programs. In each iteration:
  - The entire graph is read from the distributed file system.
  - The entire graph is transferred from Mappers to Reducers.
  - The entire graph, with updated PageRank values, is written to the distributed file system.
- On the other hand, the PageRank program does not perform irrelevant computation. In contrast to single-source shortest path, all PageRank values must be recomputed in every iteration.

PageRank in Spark

- We discuss the Spark Scala pseudo-code with RDDs. The version with DataSet is left as a voluntary challenge.
- The Spark program separates static data (the graph structure) from evolving data (the current PageRank of each vertex).
  - This leverages cached RDD partitions and avoids MapReduce's shuffling of the adjacency list data.
  - On the downside, the new PageRank data of type (vertexID, PageRank) must be joined with the graph data of type (vertexID, adjacencyList). This join on vertexID normally requires shuffling, but careful co-partitioning avoids that.
- Look at the program from the Spark 2.4.0 distribution at http://khoury.northeastern.edu/home/mirek/code/SparkPageRank.scala
  - Notice that it does not handle dangling pages.

Simplified Spark Implementation
==============================

```
// The input file contains in each line a vertex ID and its adjacency list. Function
// getAdjListPairData returns a pair of vertex ID and its adjacency list, creating a pair RDD.
graph = sc.textFile(…).map( line => getAdjListPairData(line) )

// Add code here to make sure that the graph pair RDD has a Partitioner.
// This is needed for avoiding shuffling in the join below.

// Tell Spark to try and keep this pair RDD around in memory for efficient re-use
graph.persist()

// Create the initial PageRanks using page count |V|, which can be passed through the context.
// Function mapValues ensures that the same Partitioner is used as for the graph RDD.
PR = graph.mapValues( adjList => 1.0 / |V| )

// Function extractVertices returns each vertex id m in n's adjacency list as
// (m, n's PageRank / number of n's outlinks).
for (iterationCount <- 1 to k) {      // Use Accumulator instead to detect (approximate) convergence
  PR = graph.join( PR )
        .flatMap( (n, adjList, currentPRofN) => extractVertices(adjList, currentPRofN) )
        .reduceByKey( (x, y) => (x + y) )          // Add PageRank contributions
}
```

Another Spark PageRank Program

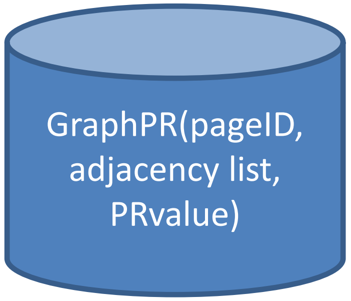
- The next example is a complete Spark Scala program from a Spark textbook. The authors set α=0.85, following the original PR paper.
- Take a moment and compare the two PR programs. What is different, what is the same?
- Notice how |V|, the number of Web pages, does not appear in the second program. Initial PR values are set to 1.0 and the updated PR is 0.15+0.85*incoming_PR_contributions.
  - Convince yourself that this is equivalent to computing the true PR values, just multiplied by |V|.

Code Fragment from *Learning Spark* by Zaharia et al.

```
// Assume that our neighbor list was saved as a Spark objectFile
val links = sc.objectFile[(String, Seq[String])]("links").partitionBy(new HashPartitioner(100))
              .persist()

// Initialize each page's rank to 1.0; since we use mapValues, the resulting RDD
// will have the same partitioner as links
var ranks = links.mapValues(v => 1.0)

// Run 10 iterations of PageRank
for (i <- 0 until 10) {
  val contributions = links.join(ranks).flatMap {
    case (pageId, (links, rank)) => links.map(dest => (dest, rank / links.size))
  }
  ranks = contributions.reduceByKey((x, y) => x + y).mapValues(v => 0.15 + 0.85*v)
}

// Write out the final ranks
ranks.saveAsTextFile("ranks")
```

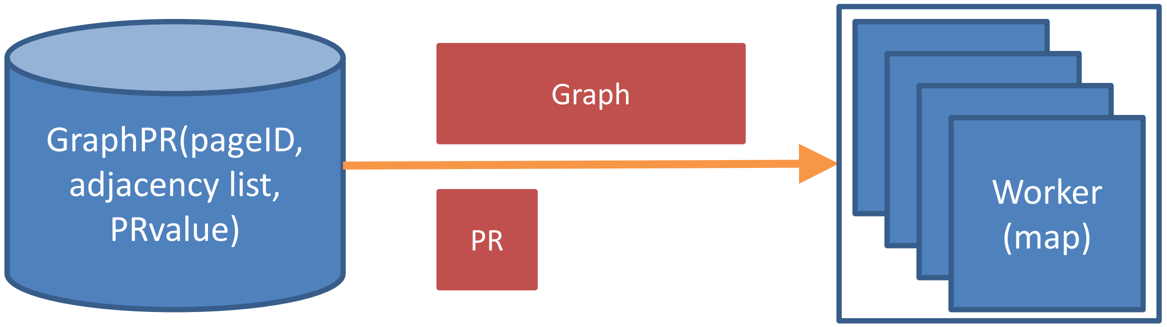
Let us look at the key difference between Spark and MapReduce.

Typical iterative job (PageRank) in Hadoop MapReduce:



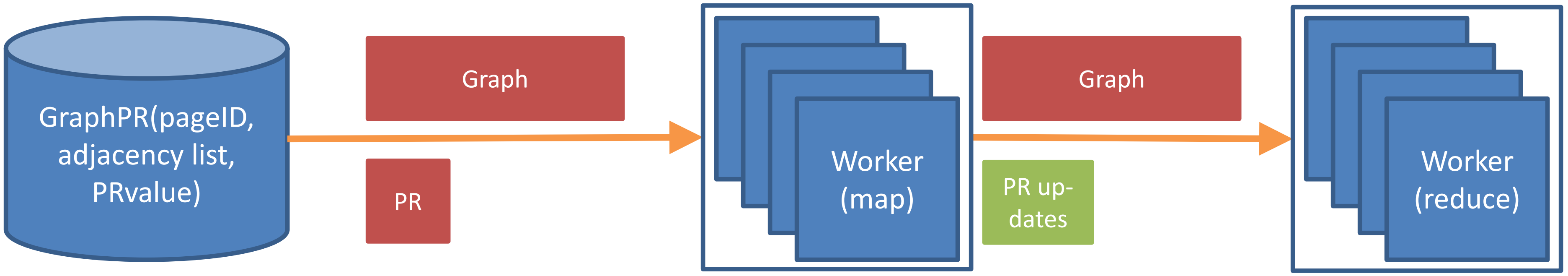
In MapReduce, each input record in GraphPR contains both the adjacency list and the PageRank value for each page. The PageRank algorithm iterates through the entire graph to update these PRvalues.

Typical iterative job (PageRank) in Hadoop MapReduce:



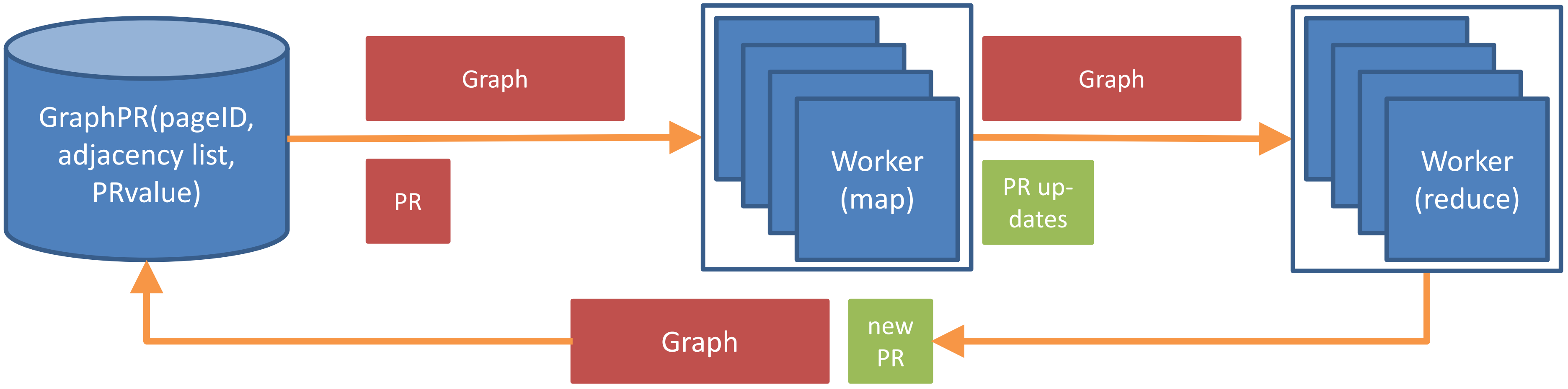
The MapReduce program first loads the Graph with its PRvalues into the Mappers. (Graph and PRvalue data are shown separately for better comparability to the Spark approach.)

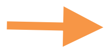
Typical iterative job (PageRank) in Hadoop MapReduce:



Each Mapper forwards graph structure and update info for the PRvalues to the corresponding Reducers.

Typical iterative job (PageRank) in Hadoop MapReduce:



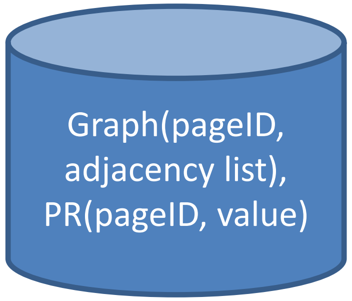


The Reducers compute the new PRvalues and emit the updated graph to the distributed file system.

This cycle repeats in the next iteration.

Spark manages data that changes in each iteration (PR: the PageRank value of a page) separate from data that does not (Graph: the graph structure, i.e., the pages and their adjacency lists of links.) This enables it to exploit in-memory data and dramatically reduce data movement.

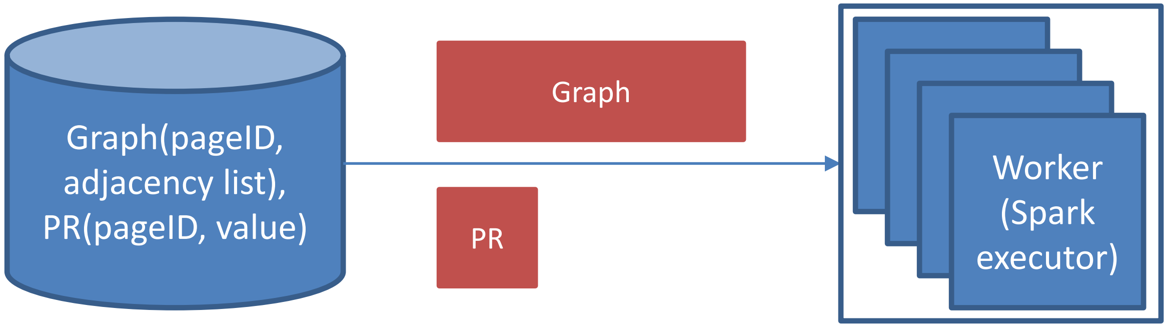
Typical iterative job (PageRank) in Spark:



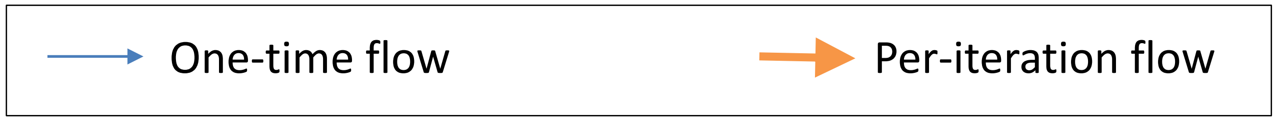
First, both Graph and PR are loaded into the Spark worker tasks. Ideally (when the graph fits into the combined memory of all tasks) this initial loading step happens only once, not in each iteration.

One-time flow
Per-iteration flow

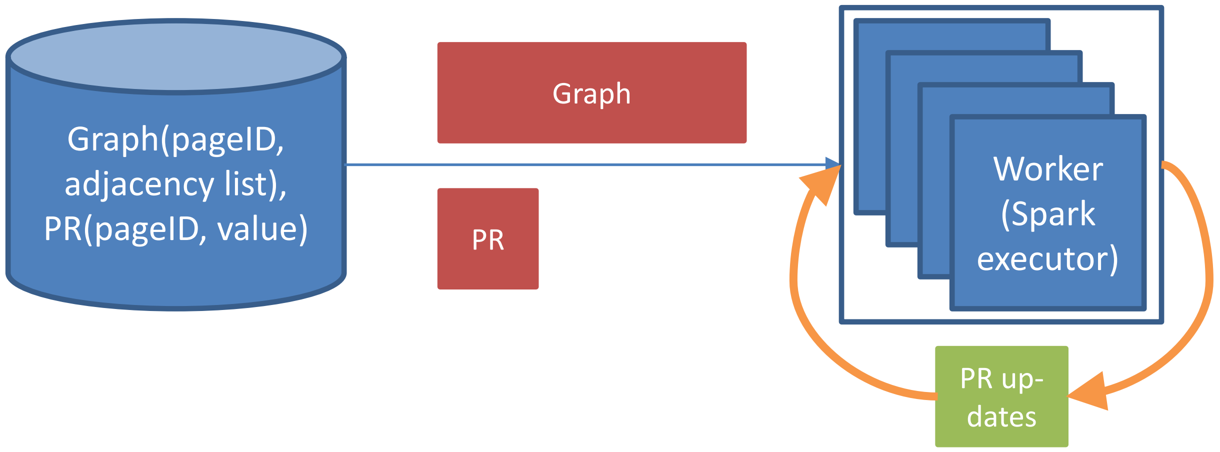
Typical iterative job (PageRank) in Spark:



Since the workers can keep the Graph data as an RDD or DataSet in memory, Graph does not need to be loaded and saved repeatedly. Instead, each iteration only passes the PR value updates to the task processing the corresponding node.



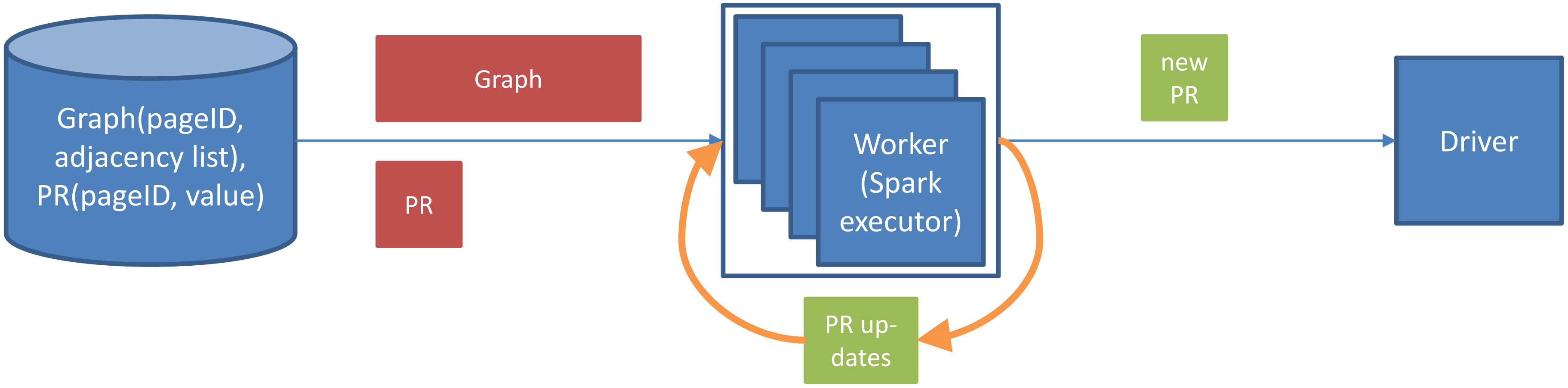
Typical iterative job (PageRank) in Spark:



In the very end, the final PR values are collected by the driver. This again is a one-time data transfer, not one happening in each iteration.

One-time flow
Per-iteration flow

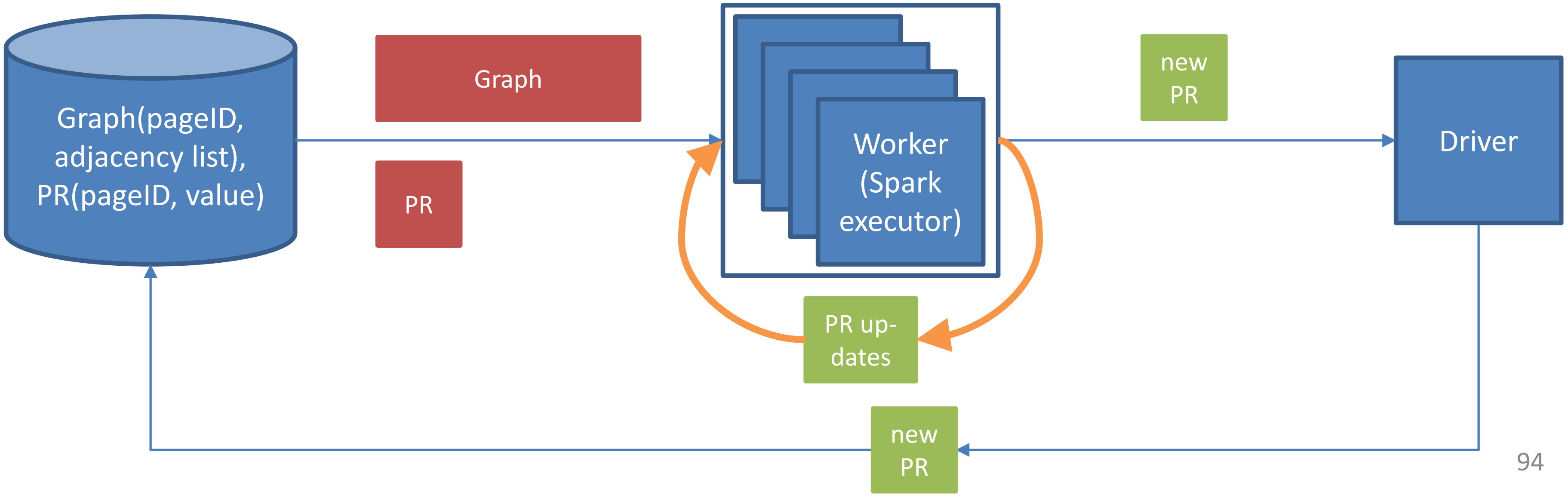
Typical iterative job (PageRank) in Spark:



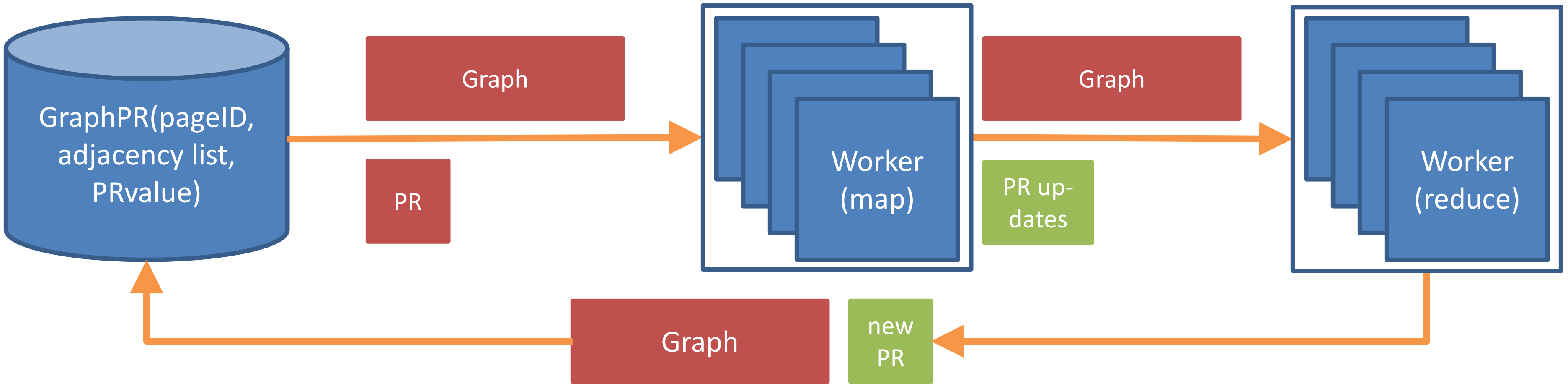
The driver then saves the final PR values to the distributed file system.

One-time flow
Per-iteration flow

Typical iterative job (PageRank) in Spark:

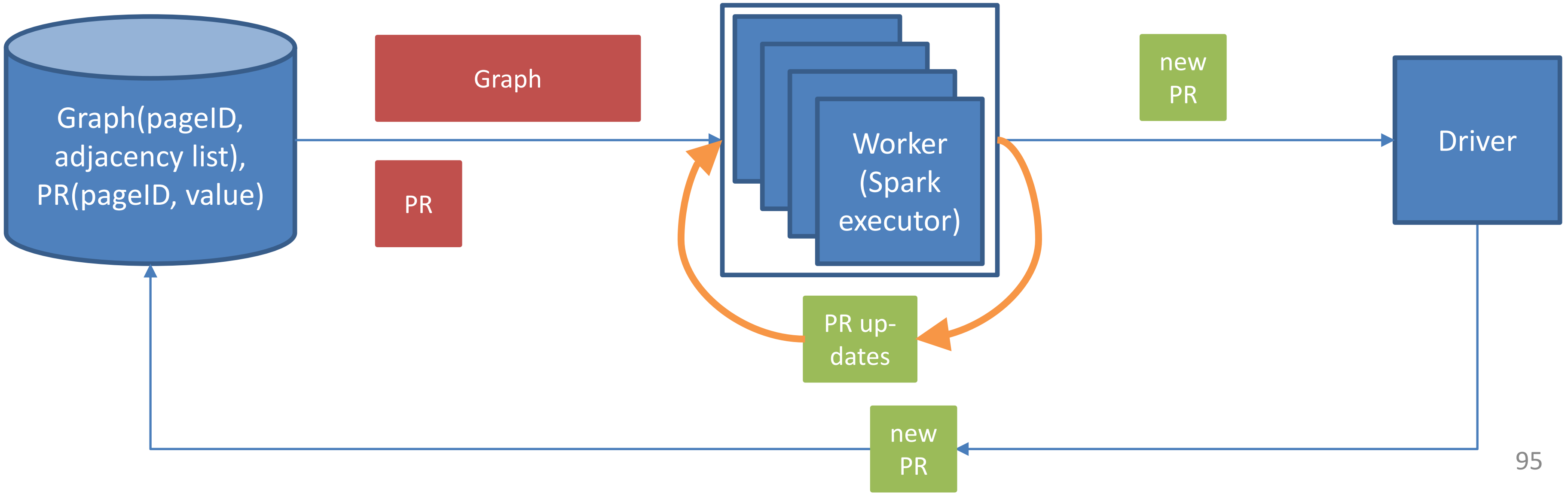


PageRank in Hadoop MapReduce:



One-time flow
Per-iteration flow

PageRank in Spark:



PageRank in a DBMS

- Assume the graph is stored in table Links(origin, destination) and the PageRank values in table PageRank(pageID, PR, outDegree), where origin, destination, and pageID are vertex IDs, and outdegree is the number of vertices in the adjacency list of vertex pageID. Then we can express the computation of an iteration as shown below.

```
newPR =  SELECT L.destination, (1-alpha)*numPages + alpha*SUM(P.PR/P.outDegree)
         FROM Links AS L, PageRank AS P
         WHERE L.origin = P.pageID
         GROUP BY L.destination

(Note that numPages = SELECT COUNT(*) FROM PageRank.)
```

Dangling Pages

- *Dangling pages*, i.e., pages that have no outgoing links, leak PageRank probability mass. Consider a dangling page x with PageRank P(x). During an iteration, P(x) is not passed along to another page and hence probability mass αP(x) disappears. Stated differently, dangling pages cause the total PageRank sum of the graph to decrease with every iteration.
- How can we address this problem? If the random surfer ends up on a dangling page, she cannot follow links and hence must perform a random jump. We model this mathematically by *conceptually* adding imaginary links from x to every page in the graph, including x itself. If we use δ to denote the total PageRank mass of dangling pages, then the corresponding formula for the PageRank of page n becomes

$$P(n) = (1-\alpha)\frac{1}{|V|} + \alpha\left(\frac{\delta}{|V|} + \sum_{m \in L(n)} \frac{P(m)}{C(m)}\right)$$

- Since PageRank values of dangling pages may change, we must compute the new value of δ in each iteration.

Dangling-Pages Solution in MapReduce

- The formula computing the new PageRank in Reduce must be modified to include the term with variable δ.
- Unfortunately, it is not possible to compute some value and also read it out in the *same* MapReduce job. We next discuss multiple possible solutions in more detail.
- Solution 1: add a separate job to each iteration to compute δ.
  - During an iteration, first execute a MapReduce job that computes δ. This is a simple global aggregation job, summing up PageRank values for all dangling pages.
  - Then pass the newly computed δ as a parameter to the modified MapReduce program that updates all PageRanks using the new formula with δ.
- Can we avoid the extra job in each iteration?

Alternative Solutions

- Solution 2: global counter.
  - We can compute δ "for free" during the Map phase. When Map processes a node with an empty adjacency list, it adds the node's PagePank to a global counter. By the end of the Map phase, this counter's value is δ. If Reducers can read the counter value, they can compute the correct new PageRank in the Reduce function. Otherwise, the global counter is read in the driver after the job completed. The driver then passes its value via the context to the next iteration's Map phase.
- Solution 3: dummy page.
  - For a dangling page n, the Map function emits (dummy, n's PageRank). The Reduce call for the dummy node then adds up all contributions. The driver reads this output and passes it into the next job's context.
- Solution 4: order inversion.
  - In Map, the page's PageRank is the old value computed in the previous iteration i. Reduce computes the new value output in iteration i+1. Hence we can apply the order inversion design pattern to send each Reducer the old PageRank values of all dangling pages, allowing it to compute δ right before executing the "normal" Reduce calls.

Solution 2 Pseudo-Code (May Not Work!)

```
map( vertex N ) {
  // Pass along the graph structure
  emit(N.id, N)

  // Compute contributions to send
  // along outgoing links
  s = N.adjacencyList.size()

  if (s>0) {
    p = N.pageRank / s
    for all m in N.adjacencyList do
      emit( m, p )
  } else // dangling page
    global_counter.add( N.PageRank )
}
```

```
// Reduce receives the vertex object for vertex m and
// the PageRank contributions for all m’s inlinks
reduce(id m, [p1, p2,…]) {
  s = 0
  M = NULL

  for all p in [p1, p2,…] do
    if isVertex(p) then
      // The vertex object was found: recover graph structure
      M = p
    else
      // A PageRank contribution from an inlink was found:
      // add it to the running sum.
      s += p

  // This only works if Reducers can read global counters
  M.pageRank = (1-α)/|V| + α·(global_counter/|V| + s)

  emit( m, M )
}
```

Solution 2 Discussion

- That program may not work in practice. By design, a global counter is initialized in the driver, updated by Map and Reduce tasks, then read out in the driver *after the job completed*.
- The above program needs to read the counter value in the Reducers. In theory this could be supported, because of the barrier between Map and Reduce phase: no Reducer can start processing its input until all Mappers have finished (and hence finished updating the counter).
- In practice, the implementation of the counter feature may not guarantee the global count to be stable and finalized until the entire job has finished. This needs to be verified based on the MapReduce documentation.
- The following program is an alternative where the counter value is passed via the job context to the next iteration.
  - Note that the output of the last iteration is not corrected for the dangling PageRank mass. To do so would require another Map-only job for cleanup.

Solution 2 Pseudo-Code

```
map( vertex N ) {
  // Update PageRank using δ from previous
  // iteration, passed from the driver via the
  // job context; δ = 0 in the first iteration
  N.pageRank += α · δ / |V|

  // Pass along the graph structure
  emit(N.id, N)

  // Compute contributions to send
  // along outgoing links
  s = N.adjacencyList.size()

  if (s>0) {
    p = N.pageRank / s
    for all m in N.adjacencyList do
      emit( m, p )
  } else // dangling page
    global_counter.add( N.PageRank )
}
```

```
// Reduce receives the vertex object for vertex m and
// the PageRank contributions for all m's inlinks
reduce(id m, [p1, p2,…]) {
  s = 0
  M = NULL

  for all p in [p1, p2,…] do
    if isVertex(p) then
      // The vertex object was found: recover graph structure
      M = p
    else
      // A PageRank contribution from an inlink was found:
      // add it to the running sum.
      s += p

  // This PageRank value does not account for dangling
  // PageRank mass yet!
  M.pageRank = (1-α)/|V| + α·s
  emit( m, M )
}
```

The driver reads the global counter's value after the job completes, passing it to the job for the next iteration.

Dangling-Pages Solution in Spark

- The Spark solutions mirror those in MapReduce.
  - We can compute the dangling PageRank mass in a separate step.
  - Instead of a global counter, we can use an Accumulator to compute the dangling mass in iteration i, then use it in iteration (i+1).
  - We can also add the dummy page to the graph RDD, accessing its PR value using `lookup`.
- All these solutions add an action: the separate computation step uses a global aggregate on the RDD, Accumulator access is an action, and so is the lookup operation.
  - This action forces a Spark-job execution for each iteration. Instead, it would be more efficient to run a single Spark job that executes all iterations.
  - Notice that the dangling-page-related action in iteration i depends on the PageRanks and the dangling PR mass in iteration (i-1). Those in turn depend on iteration (i-2), etc. This creates lineage information that is quadratic in the number of iterations j: The PR values from iteration 1 appear in the lineage of iterations 2,…, j; those from iteration 2 appear in 3,…, j; etc.
  - Challenge question: Since the PageRanks from iteration 1 are needed by the j-1 jobs triggered in later iterations, does this mean iteration 1 will be executed j times (and similarly the second iteration j-1 times, etc)? Would `persisting` the PageRanks RDD change this behavior?

Dangling-Pages Solution in a DBMS

- In SQL, we create an intermediate table with the dangling PageRank mass and rely on the optimizer to find an efficient computation strategy.
  - Note that for dangling pages n, there is no tuple with origin=n in Links. Hence we find all dangling pages by checking for page IDs that do not appear in the origin column in Links.

```
dangling = SELECT SUM(PR)
           FROM PageRank
           WHERE id NOT IN
                     (SELECT origin FROM Graph)

newPR =    SELECT L.destination,
             (1-alpha)*numPages + alpha*(dangling / numPages + SUM(P.PR/P.outDegree))
           FROM Links AS L, PageRank AS P
           WHERE L.origin = P.pageID
           GROUP BY L.destination
```

Number of Iterations

- In theory, the computation terminates when the fixpoint is reached, i.e., none of the PageRank values changes from one iteration to the next. In practice approximate numbers suffice, therefore iterations can stop as soon as all PageRank values "barely change."
  - It is often reasonable to stop when no page's PageRank changes by more than 1%. This threshold for *relative* change is computed for each page n as |new(n) – old(n)| / old(n), where old(n) and new(n) are the PageRank of n in previous and current iteration, respectively.
  - A threshold for *absolute* change |new(n) – old(n)| is difficult to set, because PageRank values can vary by multiple orders of magnitude in large graphs.
- The easiest way to check for convergence is via a global counter or Accumulator that determines for how many pages the PageRank change was greater than the threshold. The counter is updated in the Reduce function (or the corresponding Spark function), where both old and new PageRank of a page are available. The driver program then checks the counter value to decide about another iteration.
- Depending on graph size, structure, and convergence threshold, the number of iterations can vary widely. The paper that proposed PageRank reported convergence after 52 iterations for a graph with 322 million edges.

Summary

- Large graphs tend to be sparse and hence are often stored in adjacency-list or set-of-edges format. This representation enables per-vertex computation in a single round, which can pass information along outgoing edges to all direct neighbors.
  - It is possible to extend this capability by pre-computing other data structures, e.g., the list of neighbors within a certain distance, to push information directly to nodes farther away.
- Computation along incoming edges requires shuffling.
- The driver program controls execution of iterations, until a stopping condition is met.
- For some problems, the most efficient or most elegant sequential algorithms rely on a “centralized” data structure. These algorithms are difficult to parallelize directly.
- Iterative algorithms are common in practice and can be implemented easily in MapReduce and Spark. However, MapReduce’s lack of support for persistent in-memory data results in inefficiencies due to costly data transfer.

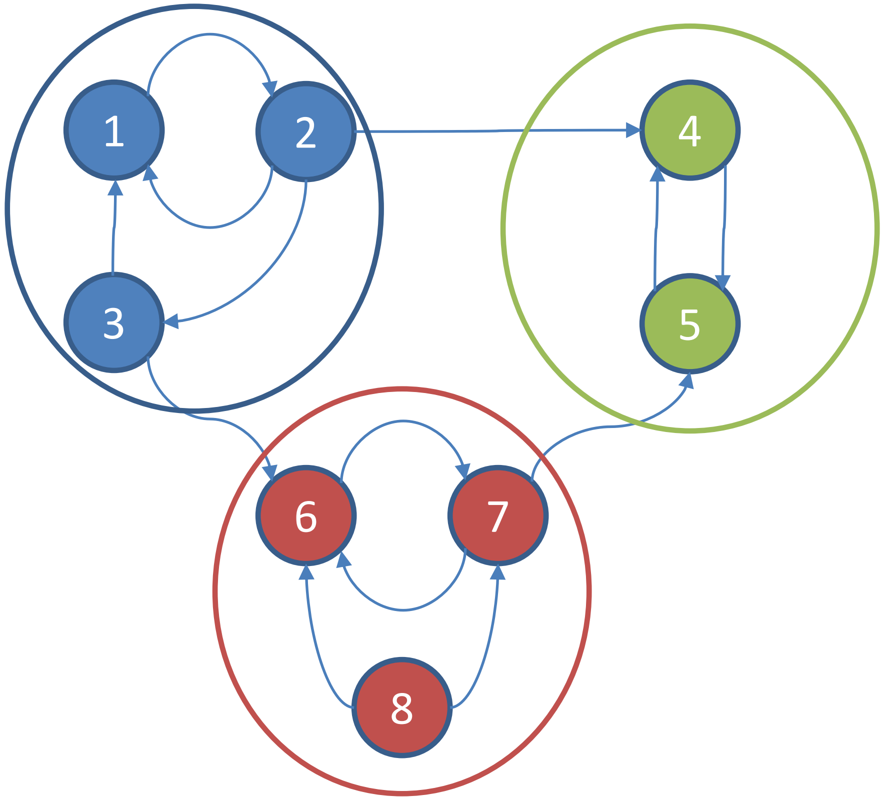
Summary (Cont.)

- Each iteration of the presented algorithms has linear time complexity in graph size, scaling to large graphs.
- For algorithms that push data along graph edges, shuffle cost in Spark depends on the number of edges “across partitions.” (We discuss this on the next pages.) A good partitioning therefore keeps densely connected regions of the graph together, cutting through sparse ones. Unfortunately, finding a balanced min-cut-style graph partitioning is hard.
- Big data raises the issue of numerical stability: In a billion-vertex graph an individual PageRank could *underflow* floating-point precision. Careful analysis is needed to decide if a number type of greater precision suffices or if a numerically more robust algorithm should be used. For instance, since log(x·y)=log(x)+log(y), one can replace multiplication by addition of log-transformed numbers.

Side-Note: Graph Partitions versus Communication

- Consider the PageRank algorithm in Spark, where the graph RDD is partitioned over the different tasks. After loading the graph for the first iteration, no further graph-data movement occurs.
- However, PageRank values still change and need to be passed along outgoing edges to be aggregated along incoming edges.
- Consider the following example partitioning, showing the adjacency lists managed by each task.

Partitions versus Communication



Graph partitions:

Partition A:
- node 1: 2
- node 2: 1, 3, 4
- node 3: 1, 6

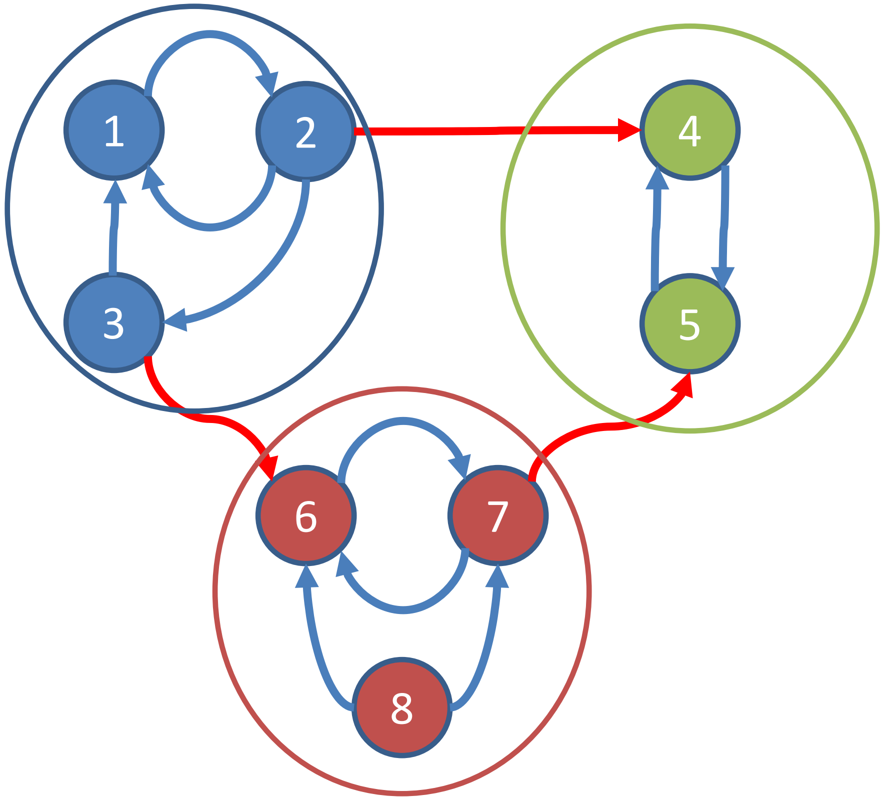
Partition B:
- node 4: 5
- node 5: 4

Partition C:
- node 6: 7
- node 7: 5, 6
- node 8: 6, 7

Partitions versus Communication

- During an iteration of the PageRank computation, each vertex sends its contributions along the outgoing edges. For destination vertices in the same partition, data transfer is local (indicated by a blue arrow on the next page).
- To avoid costly network transfer, the graph should be partitioned such that the number of edges connecting vertices in different partitions is minimized. In the example, only three edges (shown in red) require data transfer to a different task. After receiving the incoming contributions, for each vertex the new PageRank can be computed.

Efficient Iterative Processing



PageRank mass transfer:

- local transfer
- transfer to a different task

References

- S. Brin and L. Page. The anatomy of a large-scale hypertextual web search engine. Computer Networks and ISDN Systems, Elsevier, pages 107-117, 1998.
  – https://scholar.google.com/scholar?cluster=16140207188598694220&hl=en&as_sdt=0,22&as_vis=1