Linear Programming: Chapter 11
Game Theory

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Rock-Paper-Scissors

A two person game.
Rules. At the count of three declare one of:

Rock Paper Scissors

Winner Selection. Identical selection is a draw. Otherwise:

- Rock beats Scissors
- Paper beats Rock
- Scissors beats Paper

Payoff Matrix. Payoffs are *from* row player *to* column player:

$$A = \begin{array}{c|ccc|} & P & S & R \\ P & 0 & 1 & -1 \\ S & -1 & 0 & 1 \\ R & 1 & -1 & 0 \end{array}$$

Note: Any *deterministic* strategy employed by either player can be defeated systematically by the other player.

Two-Person Zero-Sum Games

Given: $m \times n$ matrix A.

- *Row player* (rowboy) selects a *strategy* $i \in \{1, ..., m\}$.
- *Col player* (colgirl) selects a *strategy* $j \in \{1, ..., n\}$.
- Rowboy pays colgirl a_{ij} dollars.

Note: The rows of A represent deterministic strategies for rowboy, while columns of A represent deterministic strategies for colgirl.

Deterministic strategies can be bad.

Randomized Strategies.

- Suppose rowboy picks i with probability y_i.
- Suppose colgirl picks j with probability x_j.

Throughout, $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T$ will denote *stochastic vectors*:

$$\begin{aligned} x_j &\geq 0, \qquad j = 1, 2, \ldots, n \\ \sum_j x_j &= 1. \end{aligned}$$

If rowboy uses random strategy y and colgirl uses x, then *expected payoff* from rowboy to colgirl is

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$

Colgirl's Analysis

Suppose colgirl were to adopt strategy x.

Then, rowboy's best defense is to use y that minimizes $y^T Ax$:

$$\min_y y^T Ax$$

And so colgirl should choose that x which maximizes these possibilities:

$$\max_x \min_y y^T Ax$$

Solving Max-Min Problems as LPs

Inner optimization is easy:

$$\min_y y^T A x = \min_i e_i^T A x$$

(e_i denotes the vector that's all zeros except for a one in the i-th position—that is, deterministic strategy i).

Note: Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

$$\max \left(\min_i \; e_i^T A x\right)$$

$$\begin{aligned} \sum_j x_j &= 1, \\ x_j &\geq 0, \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Reduction to a Linear Programming Problem

Introduce a scalar variable v representing the value of the inner minimization:

$$\max v$$

$$\begin{aligned} v &\leq e_i^T A x, \qquad i = 1, 2, \ldots, m, \\ \sum_j x_j &= 1, \\ x_j &\geq 0, \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Writing in pure matrix-vector notation:

$$\max v$$

$$\begin{aligned} ve - Ax &\leq 0 \\ e^T x &= 1 \\ x &\geq 0 \end{aligned}$$

(e denotes the vector of all ones).

Finally, in Block Matrix Form

$$\max \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{bmatrix} -A & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x \geq 0$$

$$v \;\; \textsf{free}$$

Rowboy's Perspective

Similarly, rowboy seeks y^* attaining:

$$\min_y \max_x \; y^T A x$$

which is equivalent to:

$$\begin{aligned} \min u & \\ ue - A^T y &\geq 0 \\ e^T y &= 1 \\ y &\geq 0 \end{aligned}$$

Rowboy's Problem in Block-Matrix Form

$$\min \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix}$$

$$\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \begin{matrix} \geq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y \geq 0$$

$$u \ \text{free}$$

Note: Rowboy's problem is dual to colgirl's.

MiniMax Theorem

Let x^* denote colgirl's solution to her max–min problem.
Let y^* denote rowboy's solution to his min–max problem.
Then

$$\max_x \; {y^*}^T A x = \min_y \; y^T A x^*.$$

Proof.
From *Strong Duality Theorem*, we have

$$u^* = v^*.$$

Also,

$$\begin{aligned} v^* &= \min_i e_i^T A x^* = \min_y y^T A x^* \\ u^* &= \max_j {y^*}^T A e_j = \max_x {y^*}^T A x \end{aligned}$$

QED

AMPL Model

```
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
```

AMPL Data

```
data;
set ROWS := P S R;
set COLS := P S R;
param A: P  S  R:=
      P  0  1 -2
      S -3  0  4
      R  5 -6  0
    ;

solve;
printf {j in COLS}: "    %3s %10.7f \n", j, 102*x[j];
printf {i in ROWS}: "    %3s %10.7f \n", i, 102*ineqs[i];
printf: "Value = %10.7f \n", 102*v;
```

AMPL Output

```
ampl gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective -0.1568627451
  dual objective -0.1568627451
      P 40.0000000
      S 36.0000000
      R 26.0000000
      P 62.0000000
      S 27.0000000
      R 13.0000000
Value = -16.0000000
```

Dual of Problems in General Form

Consider:

$$\begin{aligned} \max c^T x & \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Rewrite equality constraints as pairs of inequalities:

$$\begin{aligned} \max c^T x & \\ Ax &\leq b \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Put into block-matrix form:

$$\begin{array}{rcl} & \max c^T x & \\ \begin{bmatrix} A \\ -A \end{bmatrix} x & \begin{array}{c} \leq \\ \leq \end{array} & \begin{bmatrix} b \\ -b \end{bmatrix} \\ x & \geq & 0 \end{array}$$

Dual is:

$$\begin{array}{rcl} \min \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & & \\ \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & \geq & c \\ y^+, y^- & \geq & 0 \end{array}$$

Which is equivalent to:

$$\begin{array}{rcl} \min b^T(y^+ - y^-) & & \\ A^T(y^+ - y^-) & \geq & c \\ y^+, y^- & \geq & 0 \end{array}$$

Finally, letting $y = y^+ - y^-$, we get

$$\begin{array}{rcl} \min b^T y & & \\ A^T y & \geq & c \\ y & & \text{free.} \end{array}$$

Moral:

- Equality constraints $\Longrightarrow$ free variables in dual.
- Inequality constraints $\Longrightarrow$ nonnegative variables in dual.

Corollary:

- Free variables $\Longrightarrow$ equality constraints in dual.
- Nonnegative variables $\Longrightarrow$ inequality constraints in dual.

A Real-World Example

The Ultra-Conservative Investor

Consider again the historical return on investment data: We can view this as a payoff matrix in a game between *Fate* and the *Investor*.

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

Fate's Conspiracy

The columns represent pure strategies for our conservative investor.
The rows represent how history might repeat itself.
Of course, for next year (1995), Fate won't just repeat a previous year but, rather, will present some mixture of these previous years.
Likewise, the investor won't put all of her money into one asset. Instead she will put a certain fraction into each.
Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

Investor's Optimal Asset Mix:

US 3-MONTH T-BILLS	93.9
NASDAQ COMPOSITE	5.0
EAFE	1.1

Mean, old Fate's Mix:

1992	28.1
1993	7.8
1994	64.1

The value of the game is the investor's expected return: 4.10%.