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Cohen–Macaulay Ordered Sets

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1. INTRODUCTION

The purpose of this paper is to introduce a new kind of partially ordered set: the Cohen–Macaulay poset. It is now known that this concept provides some interesting connections among Algebraic Topology, Combinatorics, Commutative Algebra and Homological Algebra, and numerous individuals have contributed to the theory. Some of these are Stanley [35, 36, 37], Reisner [26], Hochster [21] and Garsia [17].

The notion of a Cohen–Macaulay poset originated in the author's thesis [2], and many of the results of this paper are also there in some form. The original motivation for introducing this concept was to provide a reasonable setting for the results of [1] and to find techniques for proving unimodality theorems. The Rank Selection Theorem (5.4) had much to do with this. At the time we referred to these posets as Folkman posets because of Folkman's work in [16]. The term "Cohen–Macaulay" was later suggested by Kempf, who pointed out the relationship with the theory of Local Cohomology as, for example, in [22].

The basic tool for proving our results is the theory of homology of diagrams on posets. Diagrams, even without homology, are related to certain purely combinatorial constructions. For an example of this see [4]. By using diagrams in more sophisticated ways one can prove some quite interesting combinatorial theorems, as was done for example in [6] using results from [3].

Although we have consistently used poset homology to prove the results in this paper, one could also prove them using ring theory methods. In a joint paper with Garsia [7], the latter approach is employed. The fact that one can define Cohen–Macaulay posets using either homology theory or ring theory is a consequence of a remarkable theorem of Reisner [26]. An "elementary" proof of this important theorem appears in [7].

One of the most dramatic applications of Cohen–Macaulay posets (or more precisely of Cohen–Macaulay *complexes*) is the proof by Stanley [35] of the

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Upper Bound Conjecture for Spheres. Although we will not discuss his result here, it was certainly one of the main motivations for the development of our theory.

This paper is organized as follows. In Section 2 we introduce the terminology and background of poset homology that we require in the paper. In the next section we define the notions of Cohen–Macaulay and of almost Cohen–Macaulay posets and prove some elementary results about them. We then go on to describe the two spectral sequences which we later use to prove our main theorems. For more detail about these spectral sequences we refer to our earlier papers [1] and [3].

In Section 4 we describe some of the many classes of posets that are known to be Cohen–Macaulay, and we describe what is known about the relationship of the Cohen–Macaulay property to other properties of posets. Semimodular lattices, (locally) semimodular posets and shellable posets are all examples of Cohen–Macaulay posets.

The next two sections describe two constructions that preserve the Cohen–Macaulay property: rank selection and fibration. A rank-selected subposet is one that is obtained by deleting all the elements of specified ranks from another poset. The rank selection theorem allows us to give some new characterizations of the Cohen–Macaulay property. Fibration, on the other hand, is a method of building a poset from smaller posets. This method is quite useful for proving that particular posets are Cohen–Macaulay.

In Section 7 we consider some operations that preserve the Cohen–Macaulay property. These include product, interval poset and replication.

In the last section we prove a "homotopy theorem" for Cohen–Macaulay posets. This result is analogous to the Tutte Homotopy Theorem in Matroid Theory. Our result may be interpreted as saying that Cohen–Macaulay posets obey a weak semi-modularity condition.

2. Posets and Diagrams

We begin by discussing some of the basic background we require. We caution the reader that, in order to avoid unwieldy notation, we have abbreviated some standard terms.

Posets

For the most part we will restrict our attention to finite posets P with the property that the elements of P may be arranged on "levels" or "ranks". To be more precise we need some auxiliary concepts. A *chain* of P is a totally ordered subset of P. We will usually write $x_1 < \cdots < x_n$ for a typical chain of P. The *rank* of a chain is the number of elements in it; thus $r(x_1 < \cdots < x_n) = n$.

More generally, the rank of P, written $r(P)$, is the rank of the longest chain of P. The *length* of P, written $l(P)$, is given by $l(P) = r(P) - 1$. The length is a more topological notion whereas the rank seems to be more combinatorial. Apparently topologists start counting at zero while combinatorialists prefer to begin at 1. We will do both. A poset P is said to be *ranked* if every maximal chain has rank $r(P)$.

Given a poset P, we will write $\hat{P}$ for the poset obtained by adjoining a *new* pair of elements to P, written $\hat{0}$, $\hat{1}$ such that $\hat{0} < x < \hat{1}$ for all $x \in P$. If we only require that $\hat{0}$ or $\hat{1}$ be adjoined, we will write $P_{\hat{0}}$ or $P^{\hat{1}}$ respectively. We use the convention that $\hat{0}$ or $\hat{1}$ is *never* an element of P. The context should indicate to which poset $\hat{0}$ or $\hat{1}$ is to be adjoined.

A subset $J \subseteq P$ will be called an *order-ideal* if for every $x \in J$, $y \leqslant x$ implies $y \in J$. The dual definition gives the concept of an *order-filter*. The order-ideal generated by a subset $S \subseteq P$ will be denoted $J(S)$ or $J_P(S)$; while $V(S) = V_P(S)$ denotes the order-filter generated by S. The special case $J(x)$ for $x \in P$ can also be denoted $(\hat{0}, x]$. If P is ranked, then so is every subset $J(x)$, and we write $r(x)$ for $r(J(x))$. The function r takes values in the set $[r(P)]$ which by definition denotes $\{1, 2,..., r(P)\}$. The length of an open interval will be denoted $l(x, y)$ instead of $l((x, y))$.

We will often use the Möbius function as defined by Rota [28]. However, we will use slightly different notation. For a poset P we write $\mu(P)$ for $\mu(\hat{0}, \hat{1})$ as computed in $\hat{P}$. For $x \in P$ we will write $\mu(x)$ or $\mu_P(x)$ for $\mu(J(x))$. Finally, for $x \leqslant y$ in P we will think of $\mu(x, y)$ as an abbreviation for $\mu((x, y))$, which fortunately coincides with the notation in [28].

Simplicial Complexes

For a finite set S, let $B(S)$ denote the poset of nonempty subsets of S. A finite *simplicial complex* is an order-ideal of $B(S)$. The minimal elements are called *vertices* and elements in general are called *simplices*. Much of what we do in the sequel may be extended routinely to simplicial complexes. As we have defined it, a simplicial complex is a special kind of poset. However, given a finite poset P, we can define the *order complex* of P, denoted $\Delta(P)$, to be the subset of $B(P)$ consisting of the nonempty chains of P. By this device one may view posets as a special kind of simplicial complex. By applying "Δ" to the theorems in this paper one can get new theorems which often generalize to simplicial complexes either in general or having some suitable additional structure.

For example, if P is ranked, then $\Delta(P)$ has the property that every maximal simplex contains the same number of vertices. A simplicial complex with this property is said to be *pure*. The rank function r on P allows one to partition the vertices of $\Delta(P)$ into disjoint subsets, each of which meets every maximal simplex just once. A simplicial complex with such a partition is said to be *completely balanced*. The notation is due to Stanley [37]. More generally he calls a simplicial complex *balanced of type* $(a_i, ..., a_m)$ if its vertex set V may be partitioned into subsets $V_1, ..., V_m$ such that each V_i meets every maximal simplex in a_i vertices. Much of our theory generalizes easily to balanced complexes.

Another poset notion that has an analog for simplicial complexes is that of an open interval. For a simplex σ of a simplicial complex Δ, the *link* of σ, written $\mathrm{link}_\Delta(\sigma)$, is the simplicial complex $\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \varnothing \text{ and } \tau \cup \sigma \in \Delta\}$. We also view Δ itself as a link: $\Delta = \mathrm{link}_\Delta(\varnothing)$. It is easy to see that $\mathrm{link}_\Delta(\sigma)$ is isomorphic to the open interval $(\sigma, \hat{1})$ in $\hat{\Delta}$ viewed as a poset. Conversely, it is easy to see that for any open interval (x, y) of $\hat{P}$, $\Delta(x, y)$ is a link of $\Delta(P)$.

We end by mentioning that every simplicial complex Δ has an associated topological space called the *geometric realization* $|\Delta|$ of Δ, and Δ is said to *triangulate* $|\Delta|$. For a definition of $|\Delta|$ see for example Spanier [30, Section 3.1].

Diagrams

Henceforth we fix a choice of a field K with respect to which all homology and related concepts will be taken, unless specified otherwise. A (commutative) *diagram* is a set of vector spaces and homomorphisms between them such that if one can traverse the diagram from one vector space to another then the composition of the homomorphisms encountered along the way does not depend on the path taken between the two vector spaces. The vector spaces are called the *stalks* of the diagram and the homomorphisms are called the *structure maps*. If the underlying pattern of a diagram D is a poset P, we say D is a *diagram on* P. The stalk at $x \in P$ will be written D_x, and the *support* of D is $\{x \in P \mid D_x \neq 0\}$. For $x \leqslant y$, the structure map goes from D_x to D_y. The structure maps will only occasionally be mentioned explicitly.

Two special kinds of diagram will be most frequently employed. The *skyscraper diagram* $K[x]$ over $x \in P$ is the diagram having one nonzero stalk, $K[x]_x \cong K$. The *constant diagram* $\tilde{K}[Q]$ on $Q \subseteq P$ is the diagram such that

$$\begin{aligned} \tilde{K}[Q]_x &\cong K \qquad \text{if} \quad x \in Q \\ &\cong 0 \qquad \text{if} \quad x \notin Q, \end{aligned}$$

and such that the structure maps are the identity on K if that is possible and the zero homomorphism otherwise. We will only use $\tilde{K}[Q]$ for Q either an order-ideal, an order-filter or an intersection of such. Note that $K[x] = \tilde{K}[\{x\}]$ is a special kind of constant diagram.

Now in algebraic topology, one associates a sequence of vector spaces $H^i(X, K)$, for $i \geqslant 0$, to any simplicial complex X. These are called the *cohomology groups* of X (with coefficients in K). The *reduced cohomology groups* of X, written $\tilde{H}^i(X, K)$, coincide with the $H^i(X, K)$ for $i > 0$. When X is nonempty, $\tilde{H}^0(X, K)$ has dimension (over K) one less than $H^0(X, K)$; while $\tilde{H}^0(\varnothing, K) = 0$ and $\tilde{H}^{-1}(\varnothing, K) \cong K$.

In a similar way, one can associate a sequence of vector spaces $H^i(D) = H^i(P, D)$, for $i \geqslant 0$, to any diagram D on P. For the definition and properties see [1, 3] and references contained in these papers. The usual cohomology and reduced cohomology groups of $\Delta(P)$ are a special case of the cohomology groups of diagrams on P.

PROPOSITION 2.1. *For any finite poset P and $i \geqslant 0$, we have*

$$H^i(P, \tilde{K}[P]) \cong H^i(\Delta(P), K),$$

and

$$H^i(P^{\hat{1}}, K[\hat{1}]) \cong \tilde{H}^{i-1}(\Delta(P), K).$$

Moreover, if $Q \subseteq P$ is an order-ideal, then

$$H^i(P, \tilde{K}[Q]) \cong H^i(\Delta(Q), K).$$

Proof. These are well-known facts, but there are proofs more or less from scratch given in [1], where the results above are Theorem 2.1, Lemma 3.1 and Lemma 1.1 respectively. ∎

The simplest poset with respect to cohomology is a one-element poset. For such a poset the *reduced* cohomology is zero identically. If P has a top or bottom element, then $\Delta(P)$ has the same cohomology as a point. In general if $\Delta(P)$ has the cohomology of a point, we say P is *acyclic*.

Next more complicated after acyclic posets are posets such that $\Delta(P)$ has nonzero reduced cohomology in exactly one dimension. We say that P is a *d-bouquet in cohomology* if $\tilde{H}^i(P, K) = 0$ for $i \neq d$. The terminology stems from the fact that if a simplicial complex Δ triangulates a set of d-dimensional spheres, all joined together at one point, then Δ has reduced cohomology only in dimension d. Note that the empty set is the unique example of a (-1)-bouquet in cohomology.

Now it is obvious by definition that $\tilde{H}^i(\Delta(P), K) = 0$ for $i > l(P)$. Thus $l(P)$ is the highest dimension possible for $\Delta(P)$ to have nontrivial reduced cohomology. We will say that P is a *bouquet* if it has nonzero cohomology at most in the highest possible dimension. In other words, P is a bouquet if and only if P is an $l(P)$-bouquet in cohomology. We add that by the Universal Coefficient Theorem, it does not matter whether we use homology or cohomology in this definition. In terms of diagrams, P is a bouquet if and only if $H^i(P^{\hat{1}}, K[\hat{1}]) = 0$ for $i \neq r(P)$.

Finally, we will write $\tilde{h}_i(P)$ for the ith reduced Betti number, $\dim_K \tilde{H}^i(\Delta(P),K)$. The *reduced Euler characteristic* of P is $\tilde{\chi}(P) = \sum_{i=0}^{l(P)} (-1)^i \tilde{h}_i(P)$. This number is also computable by counting chains of P; namely, $1 + \tilde{\chi}(P) = \sum_{i=0}^{l(P)} (-1)^i \alpha_i(P)$, where $\alpha_i(P)$ is the number of chains of P of length i. By Phillip Hall's Theorem, as observed by Rota in [28, Corollary 2 of Theorem 3], $\tilde{\chi}(P)$ coincides with $\mu(P)$.

The Filtration Spectral Sequence

One of the most useful tools in our theory is the fact that, roughly speaking, we can compute the ith cohomology of any diagram on a Cohen–Macaulay poset by looking only at the elements of ranks i, $i+1$ and $i+2$. This is analogous to the idea in topology that the ith cohomology of a polyhedron depends only on its $i+1$ skeleton. In the proposition below we write $V^{\oplus n}$, when V is a vector space, for the direct sum of n copies of V.

PROPOSITION 2.2. *Let D be a diagram on a ranked poset P of length l. Suppose that for every $x \in P$, the open interval $(\hat{0}, x)$ is a bouquet. Then there is a complex*

$$0 \to C^0 \to C^1 \to \cdots \to C^l \to 0,$$

whose cohomology is $H^*(P, D)$ and whose terms are given by

$$C^i = \bigoplus_{r(x)=i+1} (D_x)^{\oplus|\mu(x)|}.$$

Moreover, the homomorphism $C^i \to C^{i+1}$ depends only on the structure maps $D_x \to D_y$ for all $x \leqslant y$ such that $r(x) = i+1$ and $r(y) = i+2$.

Proof. This follows from [1, Corollary 4.3]. Although the result there was stated only for geometric lattices, the proof clearly generalizes to our case. ∎

The Leray Spectral Sequence

Although spectral sequences can be a formidable machine in the general case, we will need only a small part of this particular one. For an introduction to the Leray spectral sequence from a combinatorial point of view, see [3].

The idea of the Leray spectral sequence is that if $f: P \to Q$ is a map of posets which preserves the order of P, i.e. $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q, and if D is a diagram on P, then the cohomology of D may be computed by means of information on Q. For such a map we define the *fiber of f over $y \in Q$* to be the following order-filter of P:

$$f/y = \{x \in P \mid f(x) \geqslant y\}.$$

For a diagram D on P, the qth *direct image diagram* of D with respect to f, denoted $R^q f_* D$, is the diagram on Q whose stalk at $y \in Q$ is

$$(R^q f_* D)_y = H^q(f/y, D),$$

where $H^q(f/y, D)$ is the qth cohomology of the restriction of D to the fiber f/y. The structure maps of R^qf_*D are induced by the inclusions $f/y \subseteq f/y'$ whenever $y \geqslant y'$. The corresponding structure map is a map $(R^qf_*D)_{y'} = H^q(f/y', D) \to H^q(f/y, D) = (R^qf_*D)_y$, which, since $H^q(P, D)$ may be regarded as being a set of functions of a certain kind, is simply a restriction of functions.

The Leray spectral sequence says that $H^n(P, D)$ is a subspace of a quotient space of the direct sum

$$\bigoplus_{p+q=n} H^p(Q, R^qf_*D).$$

Moreover, uncer certain circumstances it will be isomorphic to this direct sum. For a more detailed description of spectral sequences we refer the reader to Cartan–Eilenberg [12]. For the special case of the Leray spectral sequence above see also [3].

3. Elementary Properties

We are now finally ready for the definition of a Cohen–Macaulay poset.

Definition. A finite poset P is said to be *Cohen–Macaulay*, abbreviated CM, if for every $x \leqslant y$ in $\hat{P}$, the open interval (x, y) is a bouquet.

We first make the trivial observation that P is CM if and only if $\hat{P}$ is CM. A less trivial observation is that CM posets are ranked.

Proposition 3.1. *If P is* CM *then P is ranked.*

Proof. There are three possibilities for an interval (x, y) of $\hat{P}$.

Case 0. $r(x, y) = 0$. In this case we say *y covers x.*

Case 1. $r(x, y) = 1$. In this case (x, y) is an *antichain* or totally unordered subset of P. Such a poset is always a bouquet.

Case 2. $r(x, y) > 1$. As the rank increases, the condition on (x, y) gets more and more subtle. But one fact is clear: if $r(x, y) > 1$, then (x, y) must be connected. This follows from the fact that $\tilde{H}^0(\Delta(x, y), K) = 0$ when $l(x, y) > 0$ in a CM poset.

We now show that the properties mentioned above imply that P is ranked. We do this by induction. We may assume that $r(P) > 1$ and that the result holds for CM posets Q such that $r(Q) < r(P)$. Let $A \subseteq P$ be the set of maximal elements of P. We partition A into subsets B, C by

$$\begin{aligned} B &= \{x \in A \mid r(\hat{0}, x) = r(P) - 1\} \\ C &= \{x \in A \mid r(\hat{0}, x) < r(P) - 1\}. \end{aligned}$$

By definition of $r(P)$, we must have $B \neq \varnothing$.

Suppose that $C \neq \varnothing$ also. Now if it were the case that $J_P(B) \cap J_P(C)$ were empty, then P would be disconnected. This cannot be the case since $l(P) > 0$. Therefore we may choose an element $y \in J(B) \cap J(C)$. Let $b \in B$, $c \in C$ be chosen so that $y \leqslant b$ and $y \leqslant c$. Choose maximal chains from y to b and from y to c. Concatenating these with a maximal chain from $\hat{0}$ to y in $P_{\hat{0}}$ gives maximal chains from $\hat{0}$ to b and from $\hat{0}$ to c in $P_{\hat{0}}$. By the inductive hypothesis, $(y, \hat{1})$ is ranked, because it is CM and has smaller rank than P. Therefore the two maximal chains just constructed have the same rank. However, by the inductive hypothesis both $(\hat{0}, b)$ and $(\hat{0}, c)$ are ranked so we have just shown that they have the same rank. But this contradicts the definitions of B and C. This contradiction implies that $C = \varnothing$ and hence, by induction, that P is ranked. ∎

Now a common technique of ours for showing that a given poset is CM will be to proceed inductively: the inductive hypothesis implying that all open intervals of $\hat{P}$ are bouquets except possibly for $P = (\hat{0}, \hat{1})$. As a result to show that P is CM, we must only prove that P is a bouquet.

Definition. A finite poset P is said to be *almost Cohen–Macaulay*, abbreviated ACM, if every open interval (x, y) of $\hat{P}$ is a bouquet, except possibly for $(x, y) = (\hat{0}, \hat{1})$.

Many of our theorems have ACM versions, and we will endeavor to point these out when possible. Here is an example.

Corollary 3.2. *If P is* ACM *and connected, then P is ranked.*

Proof. As pointed out in the proof of Proposition 3.1, we only used the fact that every open interval (x, y) of $\hat{P}$ is either an antichain or connected. ∎

Order Complexes

We extend the definition of CM poset and of ACM poset to simplicial complexes in the obvious way: a simplicial complex Δ is CM or ACM if and only if it is CM or ACM, respectively, when regarded as being a poset. Unfortunately this definition causes a dilemma: if we regard a CM poset P as being a simplicial complex via $\Delta(P)$, will it still be CM? Fortunately, it still is.

Proposition 3.3. *Let P be a poset. Then P is* CM *if and only if $\Delta(P)$ is* CM, *and similarly for* ACM.

Proof. Since every open interval (x, y) of $\hat{P}$ has the property that $\Delta(x, y)$ is isomorphic to an open interval of $\widehat{\Delta(P)}$, we see that $\Delta(P)$ being CM or ACM implies that P is also. The converse is a bit harder. The open intervals of $\widehat{\Delta(P)}$ are of two kinds:

Type 1. (σ, τ), $\tau \neq \hat{1}$. In this case $[\sigma, \tau]$ is a Boolean algebra, so (σ, τ) is isomorphic to the boundary of the standard simplex. Therefore it is trivially a bouquet.

Type 2. $(\sigma, \hat{1})$. If σ is the chain $x_1 < x_2 < \cdots < x_n$, then $(\sigma, \hat{1}) = \Delta(\hat{0}, x_1) * \Delta(x_1, x_2) * \cdots * \Delta(x_n, \hat{1})$, where "$*$" denotes the join of complexes. Let $\tilde{C}^*(\Delta)$ denote the reduced cochain complex of Δ (with coefficients in K). Then it is easy to verify that $\tilde{C}^{*-n}(\Delta(\hat{0}, x_1) * \cdots * \Delta(x_n, \hat{1}))$ is naturally isomorphic to $\tilde{C}^*(\Delta(\hat{0}, x_1)) \otimes \cdots \otimes \tilde{C}^*(\Delta(x_n, \hat{1}))$. By the (homological) Künneth formula, we conclude that $\tilde{H}^{*-n}((\sigma, \hat{1}), K) \cong \tilde{H}^*(\Delta(\hat{0}, x_1), K) \otimes \cdots \otimes \tilde{H}^*(\Delta(x_n, \hat{1}), K)$. Therefore if $\Delta(\hat{0}, x_1),..., \Delta(x_n, 1)$ are all bouquets, then so is $(\sigma, \hat{1})$.

It then follows immediately that P being CM or ACM implies that $\Delta(P)$ is also. ▮

Topological Invariance

Now if Δ is a simplicial complex, then since Δ is also a poset, we may define $\Delta(\Delta)$. The resulting simplicial complex corresponds to the barycentric subdivision of Δ. Thus Proposition 3.3 tells us that being CM is invariant under barycentric subdivision. Even more is true: CM is a topological invariant. In the following, recall that if Y is a subspace of the topological space X, then $H^i(X, Y; K)$ denotes the ***relative cohomology*** of X with respect to Y with coefficients in K.

PROPOSITION 3.4 (Munkres [24]). *A finite simplicial complex Δ is* ACM *if and only if $H^i(|\Delta|, |\Delta| - p; K) = 0$ for every point $p \in |\Delta|$ and every $i \neq \dim |\Delta|$; moreover, Δ is* CM *if and only if it is* ACM *and $\tilde{H}^i(|\Delta|, K) = 0$ for $i \neq \dim |\Delta|$.*

Proof. Suppose that Δ is ACM. Let p be in $|\Delta|$. By passing to the barycentric subdivison if necessary, using Proposition 3.2, we may assume that p is a vertex of Δ. By the excision axiom, $H^i(|\Delta|, |\Delta| - p; K) \cong H^i(\mathrm{star}_\Delta(p), \mathrm{star}_\Delta(p) - p; K)$, where $\mathrm{star}_\Delta(p)$ is the subcomplex $\{\sigma \in \Delta \mid \sigma \cup \{p\} \in \Delta\}$. Now $\mathrm{star}_\Delta(p)$ is acyclic so by the long exact sequence in reduced cohomology of the pair $(\mathrm{star}_\Delta(p), \mathrm{star}_\Delta(p) - p)$, $H^i(\mathrm{star}_\Delta(p), \mathrm{star}_\Delta(p) - p; K) \cong \tilde{H}^{i-1}(\mathrm{star}_\Delta(p) - p, K)$. Finally $|\mathrm{star}_\Delta(p) - p|$ is homotopy equivalent to $|\mathrm{link}_\Delta(p)|$. Since $\tilde{H}^i(\mathrm{link}_\Delta(p), K) = 0$ for $i \neq \dim |\Delta| - 1$, we conclude that $H^i(|\Delta|, |\Delta| - p; K) = 0$ for $i \neq \dim |\Delta|$.

Conversely, suppose that the condition on $|\Delta|$ holds. We immediately get that Δ and $\mathrm{link}_\Delta(v)$ are bouquets for any vertex v by the same reasoning as above. We also have these properties for the barycentric subdivision $\Delta(\Delta)$. Now $\mathrm{link}_{\Delta(\Delta)}(\sigma) = \Delta(\hat{0}, \sigma) * \Delta(\sigma, \hat{1})$, where the intervals are computed in $\hat{\Delta}$. Since $(\hat{0}, \sigma)$ is the poset of all proper subsets of σ, the join $\Delta(\hat{0}, \sigma) * \Delta(\sigma, \hat{1})$ is, by definition, the dth suspension of $\Delta(\sigma, \hat{1})$ where $d = \dim |\sigma|$. Thus

$\tilde{H}^i(\Delta(\sigma, \hat{1}), K) \cong \tilde{H}^{i+d}(\mathrm{link}_{\Delta(\Delta)}(\sigma), K) = 0$, for $i + d \neq \dim |\Delta| - 1$. Since $l(\sigma, \hat{1}) = \dim |\Delta| - d - 1$, we conclude that $(\sigma, \hat{1})$ is a bouquet. It follows that Δ is ACM. The theorem is now immediate. ▌

We might add parenthetically, that Quillen has a notion of a CM poset that superficially resembles our concept. Some of our theory generalizes to his case; for example, there is a version of Theorem 5.2 that holds in his theory. However, not all the theorems generalize. Munkres' Theorem above is an example of one that does not: Quillen's concept is *not* a topological invariant. For details see [25].

Möbius Functions

One of the most interesting combinatorial features of CM posets is the fact that the values of the Möbius function have a direct interpretation as the dimensions of certain homology groups. In fact, in Theorem 5.6 we essentially show that this property characterizes CM posets. Recall that $\mu(P)$ is the reduced Euler characteristic $\tilde{\chi}(P)$. Now if P is a bouquet, then $\mu(P) = \tilde{\chi}(\Delta) = (-1)^{l(P)} \tilde{h}_{l(P)}(P)$. Thus for a CM poset we have the following

PROPOSITION 3.5. *Let P be a* CM *poset. Then for $x \leqslant y$ in $\hat{P}$,*

$$\mu(x, y) = (-1)^{l(x,y)} \tilde{h}_{l(x,y)}(x, y),$$

in particular, $(-1)^{l(x,y)} \mu(x, y) \geqslant 0$.

Field Characteristic

Since there is a seemingly different concept of CM poset for every choice of a field K, it is natural to wonder how these different concepts are related to one another. By the Universal Coefficient Theorem and the fact that being a CM poset is determined by the *vanishing* of cohomology groups, it follows that replacing K by an extension field or by a subfield does not affect the CM property. Thus being CM depends only on the characteristic of K.

By another routine application of the Universal Coefficient Theorem, one can see that if P is CM over some field, then it is CM over $\mathbb{Q}$, the field of rational numbers. Moreover, if it is CM over $\mathbb{Q}$, then it is CM for all but finitely many characteristics.

Now one could define the concept of a CM poset over any ring R. Even more general notions are possible by making use of some kind of "structure diagram" other than the constant diagram $\tilde{K}$ for computing cohomology. However, we need only one other case: CM over $\mathbb{Z}$, the ring of integers. By the Universal Coefficient Theorem once more, one can show that P is CM over $\mathbb{Z}$ if and only if it is CM over *every* field.

Beyond the relationships mentioned above, there are no others among the CM conditions for the various characteristics. We summarize the above comments in this diagram of implications:

$$
\begin{array}{ccccc}
 & \nearrow & \text{CM over } \mathbb{Z}/2\mathbb{Z} & \searrow & \\
 & \nearrow & \text{CM over } \mathbb{Z}/3\mathbb{Z} & \searrow & \\
\text{CM over } \mathbb{Z} & \Rightarrow & \text{CM over } \mathbb{Z}/5\mathbb{Z} & \Rightarrow & \text{CM over } \mathbb{Q}. \\
\vdots & & \vdots & & \vdots
\end{array}
$$

CM over $\mathbb{Z} \Leftrightarrow$ for every prime p, CM over $\mathbb{Z}/p\mathbb{Z}$

CM over $\mathbb{Q} \Leftrightarrow$ for all but finitely many primes p, CM over $\mathbb{Z}/p\mathbb{Z}$

4. The Cohen-Macaulay Property

In this section we discuss how the CM property is related to other combinatorial properties of posets. The oldest result relating the CM property to another poset property is Folkman's Theorem [16, Theorem 4.1] which in current terminology says that a geometric lattice is CM (over $\mathbb{Z}$). His work easily generalizes, and for this reason Proposition 4.1 is essentially due to him.

One of the ways to prove that a poset is a bouquet is to use the Mayer–Vietoris sequence. This technique applies in particular to the class of shellable posets.

Definition. A simplical complex Δ is said to be *shellable* if

(1) Δ is pure of dimension d.

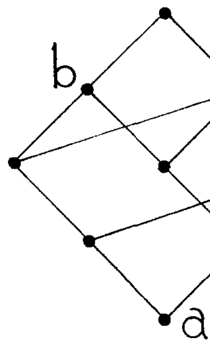
(2) The maximal simplices of Δ can be listed in some order $F_1, F_2, \ldots, F_s$ in such a way that the subcomplexes $F_n \cap (\bigcup_{i=1}^{n-1} F_i)$ are pure of dimension $d-1$ for all $n > 1$.

A poset P is said to be *shellable* if $\Delta(P)$ is a shellable complex. Any total ordering on the maximal simplices of Δ satisfying condition (2) above is called a *shelling* of Δ.

Proposition 4.1 (Folkman). *A shellable complex is* CM.

We now discuss another direction in which Folkman's work leads quite naturally. First we recall some more notation from the theory of partially ordered sets. Let P be a poset. A chain $x_0 < \cdots < x_l$ of P is said to be *saturated* if it is a maximal chain of $[x_0, x_l]$. Let $\mathrm{Cov}(P)$ be the subset of $\Delta(P)$ of saturated chains of length 1. Such chains are called *covering relations*. We say *y covers x* if $(x < y) \in \mathrm{Cov}(P)$. These are the edges of the graph one usually draws when depicting a poset. This graph is called the *Hasse diagram* of P. An (*upper*) *semi-modular lattice* L is a lattice for which if x, y both cover z in L, then $x \vee y$ covers both x and y. A finite lattice L is *geometric* if it is semimodular and every element is a supremum of elements covering the minimum element of L.

The notion of a semimodular lattice generalizes to posets. A poset P is (*upper*) *semimodular* if whenever x and y cover z in P, then there exists an element w in P which covers both x and y (Birkhoff [8, p. 39]). Now semimodularity does not imply CM, as this example shows:



This poset is not CM because (a, b) is not a bouquet, but it is easy to see that the poset is semimodular.

The problem is that semimodularity is not a local property. We define a property $\mathscr{P}$ of a poset P to be *local* if $\mathscr{P}$ is satisfied for every closed interval $[x, y]$ of P. Now CM is a local property (of $\hat{P}$) as is semimodularity for lattices. For this reason we expect that local semimodularity for posets is the proper generalization of semimodularity for lattices. In the next proposition we see that this is correct. The earliest version of this result was found by Folkman [16], who showed that geometric lattices are CM over $\mathbb{Z}$. Later Baclawski [2] and Farmer [15] independently proved that if $\hat{P}$ is locally semimodular then P is CM over $\mathbb{Z}$. Finally Björner [10, Theorem 6.1] established that in this case P is actually shellable. We now present a new proof of his result.

PROPOSITION 4.2. *A locally semimodular poset $\hat{P}$ is shellable and hence* CM.

Proof. Let $\hat{P}$ be locally semimodular. We first remark that P must be ranked. To see this let (x, y) be an open interval of $\hat{P}$ such that (x, y) is not an antichain. Let $z \in (x, y)$ be a minimal element that is not covered by y. By semimodularity, z is connected to every other minimal element of P. Therefore (x, y) is connected. Now apply the same proof as in Proposition 3.1.

Choose an ordering $x_1, \ldots, x_n$ for the minimal elements of P. We will show that we can find a shelling of P of the form $F_1, \ldots, F_{s_1}, F_{s_1+1}, \ldots, F_{s_2}, \ldots, F_{s_n}$ such that $F_1, \ldots, F_{s_i}$ is a shelling of $V(\{x_1, \ldots, x_i\})$ for all i. We assume inductively that this is possible for all locally semimodular posets $\hat{Q}$ having fewer elements than $\hat{P}$. Since $\hat{V}(\{x_1, \ldots, x_{n-1}\})$ is such a poset, we need only show that a shelling on $V(\{x_1, \ldots, x_{n-1}\})$ extends to one on P.

The elements which cover x_n are of two types: either they are in $V(\{x_1, \ldots, x_{n-1}\})$ or they are not. Let $y_1, \ldots, y_k$ be the ones of the first type, and let $y_{k+1}, \ldots, y_m$ be the ones of the second type. Now $\hat{V}(\{y_1, \ldots, y_m\})$ is a locally semimodular poset. By the inductive hypothesis we can find a shelling of it that extends one on $\hat{V}(\{y_1, \ldots, y_k\})$. Clearly this shelling induces an extension of any shelling on $\hat{V}(\{x_1, \ldots, x_{n-1}\})$ to one on $\hat{P}$.

It remains only to consider the case $n = 1$. Now in this case if we let $Q = P \backslash \{x_1\}$, then $\hat{Q}$ is a locally semimodular poset having fewer elements than $\hat{P}$. By induction it is shellable, and such a shelling induces one on P. ∎

In the special case of a geometric lattice, the result above can be stated as follows. Let $\hat{P}$ be a geometric lattice. Let $A = \{a_1, \ldots, a_m\}$ be the set of atoms of $\hat{P}$. For every maximal chain $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ of P, define its *label* to be the sequence $(b_1, \ldots, b_n)$ of atoms of $\hat{P}$ given by: b_i is the atom a_k such that k is the least integer for which $x_{i-1} \vee a_k = x_i$ (by convention $x_0 = \hat{0}$ and $x_{n+1} = \hat{1}$). Then if we order the maximal chains of P lexicographically by their labels, we get a shelling of P. This was first observed by Björner [10]. This observation has the following application.

THEOREM 4.3. *If $\hat{P}$ is a geometric lattice and if $p \in P$, then $P \backslash \{p\}$ is shellable and hence* CM.

Proof. We first note that if $x_1 \leqslant \cdots \leqslant x_n$ is maximal in $P' = P \backslash \{p\}$ then it is also maximal in P. If it were not then we could extend it to a maximal chain in P which means that for some j, $\hat{0} = x_0 \leqslant \cdots \leqslant x_{j-1} \leqslant p \leqslant x_j \leqslant \cdots \leqslant x_{n+1} = \hat{1}$ is a maximal chain of $\hat{P}$. Now the interval (x_{j-1}, x_j) cannot contain only p since $\hat{P}$ is geometric. This contradicts the maximality of $x_1 \leqslant \cdots \leqslant x_n$ in P.

Now choose an ordering $\{a_1, \ldots, a_m\}$ of the set of atoms A of $\hat{P}$ in such a way that for some k, the set of atoms below p consists of $\{a_k, \ldots, a_m\}$. In other words, choose the ordering of the atoms of A in such a way that the atoms below p are the *last* ones. We now show that with this ordering on A, the lexicographic ordering on the maximal chains of P' defines a shelling.

Let $x_1 \leqslant \cdots \leqslant x_n$ be a maximal chain of P'. Let $\{x_1 \leqslant \cdots \leqslant \hat{x}_j \leqslant \cdots \leqslant x_n \mid j \in J\}$ be the set of maximal intersections of $x_1 \leqslant \cdots \leqslant x_n$ with earlier maximal chains of P. Let y_j be chosen, for each $j \in J$, so that $x_1 \leqslant \cdots \leqslant x_{j-1} \leqslant y_j \leqslant x_{j+1} \leqslant \cdots \leqslant x_n$ is a maximal chain of P which precedes $x_1 \leqslant \cdots \leqslant x_n$. Now if none of the y_j's is equal to p, then we are done since the simplicial complex of chains contained in intersections of $x_1 \leqslant \cdots \leqslant x_n$ with preceding maximal chains of P' must in general be smaller than the complex obtained by using P instead of P'.

Therefore we may assume that for some $j \in J$ we have $y_j = p$. Let $(b_1, \ldots, b_n)$ be the label of $x_1 \leqslant \cdots \leqslant x_n$, and let $(b_1, \ldots, b_{j-1}, c_j, \ldots, c_n)$ be the label of $x_1 \leqslant \cdots \leqslant x_{j-1} \leqslant y_j \leqslant x_{j+1} \leqslant \cdots \leqslant x_n$. Now the latter maximal chain precedes the former lexicographically. So if $b_j = a_k$ and $c_j = a_l$, then $l < k$. Now by definition of the label, $x_j = x_{j-1} \vee b_j$ and $p = y_j = x_{j-1} \vee c_j$. Therefore $a_k = b_j \not\leqslant p$ and $a_l = c_j \leqslant p$. By the choice of the order of the atoms of $\hat{P}$, this tells us that $l > k$. Contradiction! The result then follows. ∎

Thus geometric lattices are "very" Cohen–Macaulay in the above sense. One consequence of Theorem 4.3 is that the canonical module of a certain ring associated to any poset P will have rank $|\mu(P)|$ when $\hat{P}$ is a geometric lattice. We refer the reader to [5] for definitions and details.

Let P be a poset. A *link* of P is a saturated chain of length 2. Write Link (P) for the set of links of P. Let $\mathscr{L} \subseteq$ Link (L) be a set of links of P. We say that a saturated chain $C = \{x_0 < \cdots < x_l\}$ is *linked by* $\mathscr{L}$ if Link $(C) \subseteq \mathscr{L}$. A ranked poset $\hat{P}$ is said to be *linkable* if there is a set $\mathscr{L}$ of links of $\hat{P}$, called a *linking* of $\hat{P}$, such that for every pair $x < y$ of elements of $\hat{P}$ there is a unique maximal chain in $[x, y]$ that is linked by $\mathscr{L}$. This concept is due to Gessel [18].

A pure simplicial complex Δ of dimension $n - 1$ is said to be a *virtual ER* (or *VER*) *complex* if there is a mapping $\phi: \Delta_{\hat{0}} \to \mathrm{Max}(\Delta) \times B([n])$, where $\mathrm{Max}(\Delta)$ is the set of maximal simplices of Δ and $B([n])$ is the Boolean algebra of subsets of $[n]$, such that:

(1) ϕ is injective,

(2) for $\sigma \in \Delta_{\hat{0}}$, if $\phi(\sigma) = (\tau, S)$, then $|S| = \dim(\sigma) + 1$,

(3) for $\tau \in \mathrm{Max}(\Delta)$, $\{S \mid (\tau, S) \in \mathrm{Im}(\phi)\}$ consists of all subsets of $[n]$ containing some fixed subset $S(\tau)$.

(4) for $\sigma \in \Delta$, if $\phi(\sigma) = (\tau, S)$, then $\sigma \subseteq \tau$.

Finally we say Δ is an *ER complex* (or a *partitionable complex*) if $\Delta_{\hat{0}}$ is a disjoint union of intervals of the form $[\sigma, \tau]$, where $\tau \in \mathrm{Max}(\Delta)$. A poset P is VER or ER if and only if $\Delta(P)$ satisfies the corresponding condition. The concepts above are due to Garsia [17].

The following gives the known relations among the poset conditions defined above.

PROPOSITION 4.4. *For a poset $\hat{P}$, the following implications hold:*

$$\begin{tikzcd}[arrows=Rightarrow]
\text{Locally semimodular} \arrow[r] & \text{Shellable} \arrow[r] \arrow[dr] & \text{CM over } \mathbb{Z} \arrow[r] & \text{CM over } \mathbb{Q} \arrow[d] \\
& \text{linkable} \arrow[r] & \text{ER} \arrow[r] & \text{VER}
\end{tikzcd}$$

Proof. The following are the nontrivial implications.

Locally semimodular $\Rightarrow$ Shellable: Björner [10]

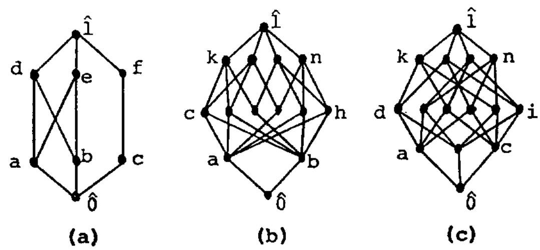
Shellable $\Rightarrow$ CM over $\mathbb{Z}$: Proposition 4.1

CM over $\mathbb{Q}$ $\Rightarrow$ VER: Garsia [17]

Shellable $\Rightarrow$ ER. This is easy. Let $\tau_1, \ldots, \tau_n$ be a shelling of $\Delta(P)$. Then for every k, $\tau_k \cap (\bigcup_{i=1}^{k-1} \tau_i)$ is the union of all subsets of τ_k which do not meet some fixed simplex $\sigma_k \subseteq \tau_k$. Therefore $\Delta(P)$ is the disjoint union of the intervals $[\sigma_k, \tau_k]$. ∎

Most of the reverse implications in Proposition 4.4 do not hold and counterexamples are easy to find. However, the precise relationship among linkability, ER and VER is still unknown. The following examples show that linkability does not imply CM even for ACM posets of low ranks.

EXAMPLE 4.5.



The following are linkings of these posets.

(a) $\mathscr{L} = \{(\hat{0}, a, d), (\hat{0}, b, e), (\hat{0}, c, f), (b, d, \hat{1}), (a, e, \hat{1}), (c, f, \hat{1})\}$

(b) $\mathscr{L} = \{(\hat{0}, a, c), (\hat{0}, a, d), (\hat{0}, a, f), (\hat{0}, a, g), (\hat{0}, a, h), (\hat{0}, b, e), (a, c, k), (a, c, l), (a, g, n), (a, h, m), (b, d, k), (b, d, l), (b, g, n), (b, h, m), (c, k, \hat{1}), (d, l, \hat{1}), (e, m, \hat{1}), (f, n, \hat{1}), (g, m, \hat{1}), (h, n, \hat{1})\}$

(c) $\mathscr{L} = \{(\hat{0}, a, e), (\hat{0}, b, d), (\hat{0}, c, f), (\hat{0}, c, g), (\hat{0}, c, h), (\hat{0}, c, i), (a, d, l), (b, e, n), (a, f, k), (a, f, m), (a, g, n), (c, g, n), (b, h, k), (c, h, k), (b, i, l), (b, i, m), (c, i, l), (c, i, m), (d, k, \hat{1}), (g, l, \hat{1}), (e, m, \hat{1}), (f, m, \hat{1}), (i, m, \hat{1}), (h, n, \hat{1})\}$.

It is easy to verify that all three posets above are ACM (when $\hat{0}$ and $\hat{1}$ are removed, of course). The first two posets are not even CM over $\mathbb{Q}$. The third poset is CM over $\mathbb{Q}$ but not over $\mathbb{Z}$. ∎

On the other hand, linkability is implied by CM in length 1. This follows from the following characterization of linkability in this case.

PROPOSITION 4.6. *Let P be a finite ranked poset of length 1. Then $\hat{P}$ is linkable if and only if the Hasse diagram of P has at most one connected component that is acyclic.*

Proof. The links of $\hat{P}$ all have either the form $\{\hat{0}, x, y\}$ or $\{x, y, \hat{1}\}$. We interpret these as directed edges (y, x) and (x, y) of the Hasse diagram G of P. Then a set L of directed edges of G is a linking of $\hat{P}$ provided:

(a) every vertex occurs exactly once as a source,

(b) there is exactly one edge directed both ways in L.

From the Hasse diagram of P we construct a bipartite graph H as follows. The vertices of H are in two classes H_1 and H_2. The elements of H_1 are the elements of P, i.e. the vertices of G. The elements of H_2 are the edges of G together with an extra vertex b. The edges of H are given as follows. If $w \in H_2$ is an edge of G, we join w to its two endpoints in H_1. If $w = b \in H_2$, we join w to every element of H_1.

We show that a matching $f\colon H_1 \to H_2$ of H gives rise to a linking $\mathscr{L}$ of $\hat{P}$ as follows.

Case 1. $b \in f(H_1)$. Then there is a unique vertex $v_0 \in H_1$ such that $f(v_0) = b$. For all $v \in H_1 \backslash \{v_0\}$, we direct the edge $f(v)$ so that v is the source. Then every vertex except v_0 is the source of a unique directed edge of $f(H_1 \backslash \{v_0\})$, and *no* edge is directed both ways, because f is a matching.

Case 1a. Suppose that v_0 is the sink of one of the directed edges above. We choose one and direct it also in the other direction. This directed edge along with the edges directed above then form the desired class $\mathscr{L}$ of links of $\hat{P}$.

Case 1b. Suppose that v_0 is not the sink of one of the directed edges above. Since P has length 1, v_0 is joined to at least one other vertex v_1 in G. This vertex is the unique source of the directed edge $f(v_1)$. If we redefine f so that $f(v_1)$ is the edge $\{v_0, v_1\}$ of G, then f is still a matching, but we are now in Case 1a.

Case 2. $b \notin f(H_1)$. Let $v_0 \in H_1$ be any element. Redefine f so that $f(v_0) = b$. Then f is still a matching, but we are now in Case 1.

Thus we must show that a matching $f\colon H_1 \to H_2$ exists. For this we use the "marriage theorem" of Philip Hall. Let $S \subseteq H_1$. We wish to show that if S' is the set of elements of H_2 joined to S, then $|S'| \geqslant |S|$. Accordingly, let $S_1, S_2, \ldots, S_n$ be the components of the subgraph generated by S in G. In a given component S_i, there must be at least $|S_i| - 1$ edges for it to be connected.

Case 1. $n = 1$. S' consists of at least $|S|$ elements because S' contains b as well as at least $|S| - 1$ edges by nature of the fact that S is connected in G.

Case 2. $n > 1$. Let S_i' be the set of all edges joined to some vertex of S_i, so that $S' = \{b\} \cup (\bigcup_i S_i')$. Now if $|S_i'| = |S_i| - 1$, then S_i forms a connected acyclic component of G. By assumption this can happen only once. Thus $|S'| = 1 + \sum_i |S_i'| \geqslant \sum_i |S_i| = |S|$, because $|S_i'| \geqslant |S_i|$ for all but possibly one S_i.

Therefore by the marriage theorem a matching $f: H_1 \to H_2$ exists and $\hat{P}$ is linkable.

Conversely if $\hat{P}$ is linkable, then a linking set $\mathscr{L}$ defines a matching $f: H_1 \to H_2$ such that $f(v_0) = b$, where v_0 is one of the vertices of the edge directed both ways by $\mathscr{L}$. Now if two components S_1, S_2 of G are acyclic then, using the notation of Case 2 above $S = S_1 \cup S_2$ has the property that $|S'| = 1 + |S_1'| + |S_2'| = 1 + |S_1| - 1 + |S_2| - 1 = |S| - 1$. This contradicts the existence of the matching f. Thus at most one component of G is a tree. ∎

Corollary 4.7. *If P has length* 1 *and is* CM, *then $\hat{P}$ is linkable.*

Although we have little evidence to support it, the examples and results above suggest that CM implies linkable.

5. Fibrations

The fibration theorem is a consequence of the Leray spectral sequence. It originally arose in an attempt to find homological analogs of Rota's theorem on the Möbius functions of posets joined by a Galois connection [27, Theorem 1] and of the Crapo complementation theorem [13, Theorem 1]. For a detailed treatment of these results see [3]. The success of these ideas prompted us to examine other contexts in which Galois connections appear naturally. One such context is the theory of fixed points in partially ordered sets as developed in [6].

Geometrically speaking, if one is given an order-preserving map $F: P \to Q$, one may regard the poset P as having been "constructed" from the fibers F/y for $y \in Q$, the "plan" for the construction being Q. One calls Q the *base* and P the *total poset* of the *fibration* F. The geometric point of view is particularly intuitive when Q is a lattice for in this case $(F/y) \cap (F/y') = F/(y \vee y')$ for $y, y' \in Q$. Thus one may think of P as being formed from the disjoint union of the fibers F/y for y minimal in Q, modulo identifications of the subfibers $F/(y \vee y')$ for y, y' minimal in Q.

Let us begin with a simple example. Roughly speaking, this proposition says that fibrations preserve acyclicity in all cases.

PROPOSITION 5.1. *Let $F: P \to Q$ be an order-preserving map of posets. Assume that:*

(1) *Q is acyclic,*

(2) *for every $y \in Q$, F/y is acyclic.*

Then P is acyclic.

Proof. This result is essentially well-known to topologists, but we give a proof in detail to illustrate the technique. We will be less detailed later. Extend F to an order-preserving may $\hat{F}: P^{\hat{1}} \to Q^{\hat{1}}$ by defining $\hat{F}(\hat{1})$ to be $\hat{1}$. We compute $H^*(P^{\hat{1}}, K[\hat{1}])$ by using the Leray spectral sequence:

$$H^p(Q^{\hat{1}}, R^q\hat{F}_*K[\hat{1}]) \Rightarrow H^n(P^{\hat{1}}, K[\hat{1}]).$$

We first compute the stalks of the diagrams $R^q\hat{F}_*K[\hat{1}]$. Let y be in $Q^{\hat{1}}$. Then

$$\begin{aligned}(R^q\hat{F}_*K[\hat{1}])_y &= H^q(\hat{F}/y, K[\hat{1}]) \\ &\cong \tilde{H}^{q-1}(F/y, K).\end{aligned}$$

By assumption (2), this vanishes for all $y \in Q$. Thus the only nonzero stalk is over $\hat{1}$, and this occurs only for $q = 0$. Therefore $R^q\hat{F}_*K[\hat{1}] = 0$ for $q > 0$ and $R^0\hat{F}_*K[\hat{1}] = K[\hat{1}]$. We now apply assumption (1) to compute

$$H^p(Q^{\hat{1}}, R^0\hat{F}_*K[\hat{1}]) \cong \tilde{H}^{p-1}(Q, K) = 0,$$

for all p. Thus by the Leray spectral sequence, $H^n(P^{\hat{1}}, K[\hat{1}]) \cong \tilde{H}^{n-1}(P, K) = 0$ for all n. The result therefore follows. ∎

We now come to the main result of this section. This says that if the fibers have the correct ranks then fibrations preserve the CM property.

FIBRATION THEOREM 5.2. *Let $F: P \to Q$ be an order-preserving map of finite ranked posets. Assume that:*

(1) $r(P) \geq r(Q)$,

(2) *Q is* ACM,

(3) *for every $y \in Q$, F/y is* CM,

(4) *for every $y \in Q$, $r(P) - r(F/y) = r(y) - 1$.*

Then P is ACM.

Furthermore P is CM *if and only if one of the following conditions holds*

(a) *$r(P) = r(Q)$ and Q is* CM, *or*

(b) *Q is acyclic.*

Proof. We remark that if condition (3) holds then conditions (1) and (4) are together equivalent to assuming that for every $y \in Q$, F/y is nonempty and its minimal elements all have rank $r(y)$. As a result, conditions (1)–(4) imply that F is rank-decreasing: if x is in P then $r(x) \geqslant r(F(x))$, because x is in $F/F(x)$ and the rank of any minimal element of $F/F(x)$ is $r(F(x))$.

Let x be in P. Then we have $x \in F/F(x)$ as just noted so $(x, \hat{1})$ is an open interval of $\widehat{F/F(x)}$. By condition (3), $(x, \hat{1})$ is CM. We next show that $(\hat{0}, x)$ is CM also. This is a bit more difficult and requires that we apply the theorem inductively.

More precisely, let P' be $(\hat{0}, x)$ and define Q' to be $(\hat{0}, F(x))$ if $F(x) \notin F(P')$ and to be $(\hat{0}, F(x)]$ otherwise. Let $G: P' \to Q'$ be the restriction of F. We show that conditions (1)–(4) and one of (a) or (b) hold for G.

The fact that F is rank-decreasing implies that $r(P') = r(x) - 1 \geqslant r(F(x)) - 1 = r(Q')$ provided that $F(x) \notin F(P')$. On the other hand, if $F(x) \in F(P')$, say $F(x) = F(x')$ where $x' \in P'$, then $r(P') \geqslant r(x') \geqslant r(F(x')) = r(F(x)) = r(Q')$. Thus condition (1) holds for G.

That condition (2) holds for G is trivial. We therefore consider condition (3). Let y be in Q'. Then we have $y \leqslant F(x)$ so that x is in F/y. It follows that G/y coincides with the open interval $(\hat{0}, x)$ as computed in $\widehat{F/y}$. Since F/y is CM by condition (3), we conclude that G/y is CM also. Hence G satisfies condition (3).

Since the rank of an element in P' or in Q' coincides with the corresponding rank in P or Q and since a minimal element of G/y is automatically minimal in F/y, we see that G also satisfies condition (4). This leaves conditions (a) and (b). If $Q' = (\hat{0}, F(x)]$, then G satisfies (b). We may therefore assume that $F(x) \notin F(P')$. It follows that x is a minimal element of $F/F(x)$. By condition (4) for F, $r(x) = r(F(x))$. Hence $r(P') = r(Q')$. By condition (2), Q' is CM, so G satisfies condition (a). Hence G necessarily satisfies one of (a) or (b).

Therefore, by induction on the size of P, we conclude that $(\hat{0}, x)$ is CM. Combining this with our earlier result, we find that P is ACM.

It remains to consider when P can be CM. We proceed as in Proposition 6.1. Let y be in $Q^{\hat{1}}$. By conditions (2) and (3), $(R^q\hat{F}_*K[\hat{1}])_y = H^q(\hat{F}/y, K[\hat{1}]) = 0$ for all q except possibly for $q = 0$ or $q = r(F/y) = r(P) - r(y) + 1$. $(R^0\hat{F}_*K[\hat{1}])_y$ is nonzero only when F/y is empty. This occurs only for $y = \hat{1}$ so $R^0\hat{F}_*K[\hat{1}] = K[\hat{1}]$ on $Q^{\hat{1}}$. Similarly, for $q > 0$ $R^q\hat{F}_*K[\hat{1}]$ is supported on elements such that $q = r(P) - r(y) + 1$, i.e. on elements of rank $r(P) - q + 1$. Therefore $H^p(Q^{\hat{1}}, R^q\hat{F}_*K[\hat{1}]) = 0$ except possibly when $p = r(P) - q$ or when $q = 0$.

The Leray spectral sequence then implies that $H^n(P^{\hat{1}}, K[\hat{1}]) \simeq H^n(Q^{\hat{1}}, K[\hat{1}])$ for $n < r(P)$, since the only terms $H^p(Q^{\hat{1}}, R^q\hat{F}_*K[\hat{1}])$ which are nonzero and have $n = p + q < r(P)$ are those for which $q = 0$, and we know in this case that $R^0\hat{F}_*K[\hat{1}] = K[\hat{1}]$. If $r(P) > r(Q)$ then P is CM if and only if $H^n(Q^{\hat{1}}, K[\hat{1}]) = 0$ for all n, i.e. Q is acyclic. If $r(P) = r(Q)$ then P is CM if and only if Q is also. The result therefore follows. ∎

6. Rank Selection

One of the most important properties of CM posets is the fact that they are preserved under rank selection: any subposet of a CM poset, obtained by deleting all elements from certain ranks, is also CM. The power of this fact is illustrated by Theorem 6.6, which characterizes CM posets in terms of this property combined with the one in Proposition 3.5.

Definition. Let P be a ranked poset of rank n. Let $S \subseteq [n]$ be a set of ranks of P. The *rank-selected subposet with respect to* S is defined by

$$P_S = \{x \in P \mid r(x) \in S\}.$$

The key to the rank selection theorem is the following immediate consequence of the filtration spectral sequence (Proposition 2.2).

Rank Selection Lemma 6.1. *Let P be an* ACM *poset and D a diagram on P. If D is supported on P_S then $H^{*-1}(P, D)$ is also supported on S, i.e. if $D_x = 0$ for every x such that $r(x) \notin S$, then $H^{i-1}(P, D) = 0$ for $i \notin S$.*

We begin with a theorem which gives a relatively general prescription for extracting ACM posets from others. The key to this process is a condition on the "distance" between elements of the ambient poset and the subposet. More precisely, let Q be an order-ideal of P. The *distance* $d_Q(x)$ of $x \in P$ from Q is the length of the smallest chain from x to an element of Q. If P is ranked, we can give a simple formula for the distance: $d_Q(x) = r(J(x)) - r(J(x) \cap Q)$. We extend this function to $\hat{P}$ by defining $d_Q(\hat{1})$ to be $r(P) - r(Q) + 1$ and $d_Q(\hat{0})$ to be 0.

Ideal Bouquet Theorem 6.2. *Let Q be an order-ideal of a* CM *poset P such that $d_Q\colon \hat{P} \to \mathbb{Z}$ is order-preserving. Then Q is a bouquet.*

Proof. We first consider the special case for which P has a maximum element x_1. If $x_1 \in Q$, then we are done. If not, we replace P by $P' = P \backslash \{x_1\}$. It is easy to verify that the hypotheses of the lemma still hold for P' and $Q' = Q$ since $d_{Q'}(\hat{1}) = d_Q(x_1)$ and $d_{Q'}(x) = d_Q(x)$ for $x \in P'$. Therefore we may henceforth assume that P does not have a maximum element.

We next observe that if P were replaced by $J(x)$ and Q by $J(x) \cap Q$, then the hypotheses of the lemma still hold. To see this simply note that $d_{J(x) \cap Q}(y) = r(J(y)) - r(J(y) \cap Q) = d_Q(y)$ for $y \in J(x)$ and $d_{J(x) \cap Q}(\hat{1}) = d_Q(x) + 1$. Now $J(x)$ is CM. Since $J(x) \neq P$ by the assumption above, we may use induction on the number of elements of P. Therefore we may assume that $J(x) \cap Q$ is a bouquet for every $x \in P$.

We now compute $H^*(Q, K)$ by using the Leray spectral sequence. To do this we must dualize P and the notation of the theorem. For example, Q is now an order-filter of P. Let $f: Q^{\hat{1}} \to P^{\hat{1}}$ be the inclusion map. The Leray spectral sequence for the diagram $K[\hat{1}]$ on $Q^{\hat{1}}$ is:

$$H^p(P^{\hat{1}}, R^q f_* K[\hat{1}]) \Rightarrow H^n(Q^{\hat{1}}, K[\hat{1}]).$$

The stalk at $x \in P^{\hat{1}}$ of $R^q f_* K[\hat{1}]$ is given by

$$H^q(f/x, K[\hat{1}]) \cong \tilde{H}^{q-1}(V(x) \cap Q, K).$$

Since $V(x) \cap Q$ is a bouquet, we see that $\tilde{H}^{q-1}(V(x) \cap Q, K) = 0$ for $q - 1 \neq l(V(x) \cap Q)$ and hence that $R^q f_* K[\hat{1}]$ is supported on $\{x \in P^{\hat{1}} \mid r(V(x) \cap Q) = q\}$.

We now use the condition on d_Q. This condition implies that for $x \in P$, $r(V(x)) - r(V(x) \cap Q) = d_Q(x) \leqslant d_Q(\hat{0}) = r(P) - r(Q) + 1$. Rearranging, we find that $r(P) - r(V(x)) \geqslant r(Q) - r(V(x) \cap Q) - 1$. The left side of this inequality is $r(x) - 1$. Therefore for every $x \in P$,

$$r(x) \geqslant r(Q) - r(V(x) \cap Q).$$

This inequality also holds for $x = \hat{1}$. Therefore $R^q f_* K[\hat{1}]$ is supported on $\{x \in P^{\hat{1}} \mid r(x) \geqslant r(Q) - q\}$.

Since P is CM, $P^{\hat{1}}$ is also; hence the filtration lemma implies that $H^p(P^{\hat{1}}, R^q f_* K[\hat{1}])$ vanishes for $p < r(Q) - q$. By the Leray spectral sequence, $H^n(Q^{\hat{1}}, K[\hat{1}])$ must also vanish for $n < r(Q)$. This is precisely the condition that Q be a bouquet. ∎

We now give the ACM version of the Ideal Bouquet Theorem.

THEOREM 6.3. *Let Q be an order-ideal in an* ACM *poset P. If $d_Q: \hat{P} \to \mathbb{Z}$ is order-preserving, then $H^i(P, K) \cong H^i(Q, K)$ for $i < l(Q) - 1$. If, moreover, d_Q satisfies $d_Q(x) \leqslant r(P) - r(Q)$ for all $x \in P$, then $H^i(P, K) \cong H^i(Q, K)$ for $i < l(Q)$.*

Proof. We proceed as in the proof of Theorem 6.2. In particular this requires us to dualize P and the notation of the theorem as we did in 6.2. Now if P has a maximum element, then P is CM and we are done by Theorem 6.2. Similarly if P were replaced by $J(x)$ and Q by $J(x) \cap Q$ then since $J(x)$ is CM we would again be reduced to Theorem 6.2. Therefore in the Leray spectral sequence computation we may again conclude that the support of $R^q f_* K[\hat{1}]$ is contained in $\{x \in P^{\hat{1}} \mid r(x) \geqslant r(Q) - q\}$.

Now the only direct image that has a nonzero stalk on $\hat{1}$ is $R^0 f_* K[\hat{1}]$. Since P is ACM, this means we may apply the filtration lemma to $R^q f_* K[\hat{1}]$ when $q > 0$. Therefore $H^p(P^{\hat{1}}, R^q f_* K[\hat{1}]) = 0$ for $p < r(Q) - q$ as before. However, since $P^{\hat{1}}$ need not be ACM, we cannot do the same for $R^0 f_* K[\hat{1}]$; but we can assert by the Leray spectral sequence that

$$H^n(Q^{\hat{1}}, K[\hat{1}]) \cong H^n(P^{\hat{1}}, R^0 f_* K[\hat{1}]), \qquad \text{for} \quad n < r(Q),$$

because of what we know about $R^q f_* K[\hat{1}]$ for $q > 0$. Now $R^0 f_* K[\hat{1}]$ is easily seen to be the constant diagram $\tilde{K}[U^{\hat{1}}]$, where $U = \{x \in P \mid V(x) \cap Q = \varnothing\}$. It is easy to see that $K[\hat{1}]$ is a subdiagram of $\tilde{K}[U^{\hat{1}}]$. Therefore we have a short exact sequence of diagrams

$$0 \to K[\hat{1}] \to \tilde{K}[U^{\hat{1}}] \to \tilde{K}[U] \to 0. \tag{$*$}$$

We now consider the condition on d_Q. The fact that $d_Q(x) \leqslant d_Q(\hat{0})$ for $x \in P$ tells us, as in the proof of 6.2, that $r(x) \geqslant r(Q) - r(V(x) \cap Q)$ and hence if x is in U then $r(x) \geqslant r(Q)$. Moreover, if we assume the stronger condition $d_Q(x) \leqslant r(P) - r(Q)$ holds, then $r(x) > r(Q) - r(V(x) \cap Q)$ holds and hence if x is in U then $r(x) > r(Q)$. Since P is ACM, we may conclude that $H^n(P, \tilde{K}[U]) = 0$ for all $n < r(Q) - 1$ in the former case and for $n \leqslant r(Q) - 1$ in the latter. Applying this to the long exact sequence of $(*)$, we find that

$$H^n(P, \tilde{K}[\hat{1}]) \cong H^n(P, \tilde{K}[U^{\hat{1}}])$$

for $n < l(Q)$ or $n \leqslant l(Q)$ as the case may be. Now the left-hand side above is isomorphic to $\tilde{H}^{n-1}(P, K)$ and the right-hand side is isomorphic to $H^n(Q^{\hat{1}}, K[\hat{1}]) \cong \tilde{H}^{n-1}(Q, K)$ for $n \leqslant l(Q)$ by our earlier computation. The result therefore follows. ∎

We now come to the main result.

Rank Selection Theorem 6.4. *If P is* CM *of rank n and if $S \subsetneq [n]$ is any set of ranks, then P_S is also* CM.

Proof. By induction we may assume that P_S is almost Cohen–Macaulay. Now $\Delta(P_S)$ is an order-ideal in $\Delta(P)$, and we have $d_{\Delta(P_S)}(\sigma) = |\sigma| - |\sigma_S|$, where $\sigma_S = \{x \in \sigma \mid r(x) \in S\}$. Thus $d_{\Delta(P_S)}$ either remains constant or increases by one as we adjoin a new element to the chain σ. By the Ideal Bouquet Theorem 6.2 we are done. ∎

With exactly the same proof one can show that any rank-selected subcomplex of a balanced Cohen–Macaulay complex is also Cohen–Macaulay (see Section 2 for notation).

As expected there is an ACM version also.

Theorem 6.5. *If P is* ACM *of rank n and if $S \subsetneq [n]$ is any set of ranks, then P_S is also* ACM *and $H^i(P_S, K) \cong H^i(P, K)$ for $i < l(P_S)$.*

Proof. That P_S is ACM follows from Theorem 6.4. Now the maximum value taken by $d_{\Delta(P_S)}(\sigma)$ is clearly $n - |S| = r(P) - r(P_S)$ so by Theorem 6.3, the second part follows also. ∎

We now show that the two rank selection theorems above allow us to exhibit a variety of useful equivalent formulations of Cohen–Macaulayness for posets. In the following, $\mu_S(x, y)$ denotes $\mu((x, y)_S)$, and in the special case $S = [s]$ for some integer s, we abbreviate $\mu_S(x, y)$ to $\mu_s(x, y)$ and $(x, y)_S$ to $(x, y)_s$.

THEOREM 6.6. *Let P be a finite ranked poset. The following properties of P are equivalent.*

(a) *P is Cohen–Macaulay,*

(b) *for every $x \leqslant y$ in $\hat{P}$ and every $S \subseteq [r(x, y)]$,*

$$(-1)^{|S|-1} \mu_S(x, y) = \tilde{h}_{|S|-1}((x, y)_S, K);$$

(c) *for every $x \leqslant y$ in $\hat{P}$ and $s \leqslant r(x, y)$,*

$$(-1)^{s-1} \mu_s(x, y) \geqslant \tilde{h}_{s-1}((x, y)_s, K);$$

(d) *for every $x \leqslant y$ in $\hat{P}$ and $s \leqslant r(x, y)$,*

$$\tilde{h}_{s-2}((x, y)_s, K) = 0.$$

Proof. (a) ⇒ (b) follows from Theorem 6.4 while (b) ⇒ (c) is trivial. We assume inductively that the theorem holds for all posets having fewer elements than P. Let P' be $P_{[r(P)-1]}$, i.e., delete the top rank of P. Now if any one of (b) through (d) holds for P, then the same is true of P' and of any open subinterval of $\hat{P}$. Therefore we may assume that P is ACM and that P' is CM.

By an easy application of the filtration lemma, we have that $\tilde{H}^i(P, K) \cong \tilde{H}^i(P', K)$ for $i < l(P) - 1$. Therefore $\tilde{H}^i(P, K) = 0$ for $i < l(P) - 1$. It is then immediate that (d) ⇒ (a). It remains to show that (c) ⇒ (d). Now $\mu(P)$ is the reduced Euler characteristic of P. Hence

$$\mu(P) = (-1)^{l(P)} \tilde{h}_{l(P)}(P, K) + (-1)^{l(P)-1} \tilde{h}_{l(P)-1}(P, K).$$

It is now immediate that (c) ⇒ (d) and the result follows. ∎

7. Cohen-Macaulay-Preserving Operations

In this section we discuss some of the more traditional operations, with respect to the CM property. We delayed the discussion of these operations until now in order to make use of the techniques of Sections 5 and 6, thereby illustrating their use.

Let P and Q be posets. The *order-dual*, denoted P^*, of P is the poset obtained by reversing the order of P. The *product of* P and Q, denoted $P \times Q$, is the cartesian product of P and Q with order given by $(x, y) \leqslant (x', y')$ if and only if $x \leqslant x'$ and $y \leqslant y'$. The *interval poset* of P, denoted $\mathrm{Int}(P)$, is the poset of closed intervals of P, ordered by inclusion: $[x, y] \leqslant [x', y']$ if and only if $x' \leqslant x$ and $y \leqslant y'$. In other words, $\mathrm{Int}(P)$ is given the induced order as a subset of $P^* \times P$.

Now it is trivial that P is CM or ACM if and only if P^* is CM or ACM; moreover $H^*(P, K) \cong H^*(P^*, K)$. This follows from Proposition 3.3 and the fact that $\Delta(P) = \Delta(P^*)$. We would like to prove similar statements for $P \times Q$ and $\mathrm{Int}(P)$.

THEOREM 7.1. *Let P and Q be nonempty* ACM *posets. Then $P \times Q$ is also* ACM, *and $P \times Q$ is* CM *if and only if both P and Q are acyclic or both are antichains.*

Proof. Let P and Q be ACM posets. Let $(x, y) \in P \times Q$. Then $J((x, y)) = J(x) \times J(y)$ and $V((x, y)) = V(x) \times V(y)$. If we assume the theorem is true for posets P' and Q' whose product $P' \times Q'$ has fewer elements than $P \times Q$, then we may assume that $P \times Q$ is ACM except when both P and Q have a minimum or both have a maximum element.

First suppose that both P and Q have minimum and maximum elements. Then $P \times Q$ has these also. Write R for the poset such that $\hat{R} = P \times Q$. As above, we can easily see that R is ACM.

Let P' and Q' be such that $P'_{\hat{0}} = P$, $Q'_{\hat{0}} = Q$. Then $P' \times Q$ and $P \times Q'$ are order-filters of $R^{\hat{1}}$ satisfying $(P' \times Q) \cup (P \times Q') = R^{\hat{1}}$ and $(P' \times Q) \cap (P \times Q') = P' \times Q'$. Now $P' \times Q$, $P \times Q'$ and $P' \times Q'$ are all CM by the second part of the theorem and our inductive hypothesis. A routine application of the Mayer–Vietoris sequence for the diagram $K[\hat{1}]$ gives that R is a bouquet. Thus R is CM, and so $P \times Q = \hat{R}$ is also.

The next case to consider is that P has a maximum, but no minimum element, and Q has a maximum element. Let $f\colon P \times Q \to P$ be given by $f(x, y) = x$. The fibers of f have the form $f/x = V(x) \times Q$. By the case considered above, these are all CM. Since the other conditions of Theorem 5.2 are satisfied, we conclude that $P \times Q$ is CM. The two cases just considered now give that $P \times Q$ is ACM in all cases.

The last statement is an easy consequence of the Eilenberg–Zilber Theorem [30, Theorem 5.3.6], the Künneth formula and the fact that $\Delta(P \times Q)$ triangulates $|\Delta(P)| \times |\Delta(Q)|$. ∎

THEOREM 7.2. *A poset P is* CM *or* ACM *if and only if* $\mathrm{Int}(P)$ *is also. Moreover, for an arbitrary poset P, $H^*(P, K) \cong H^*(\mathrm{Int}(P), K)$.*

Proof. We begin with the second statement. Define an order-preserving map $f\colon \Delta(P)^* \to \operatorname{Int}(P)^*$ by

$$f(x_0 < \cdots < x_l) = [x_0, x_l].$$

The fibers of f are given by

$$f/[a, b] = \{x_0 < \cdots < x_l \mid [x_0, x_l] \subseteq [a, b]\} = \Delta([a, b]).$$

Therefore, the fibers of f are acyclic. By the Leray spectral sequence for f and the constant diagram, we have

$$H^*(P, K) \cong H^*(\Delta(P), K) \cong H^*(\operatorname{Int}(P), K).$$

If we show that P is ACM if and only if $\operatorname{Int}(P)$ is so, then the corresponding result for CM will follow from the isomorphism above.

Assume that P is ACM. Let $[x, y]$ be in $\operatorname{Int}(P)$. Then $V([x, y]) \cong J(x) \times V(y)$ and $J([x, y]) = \operatorname{Int}([x, y])$. The former is CM by Theorem 7.1; the latter is CM by the usual induction, except when P has both a minimum and a maximum element.

Accordingly, we may suppose that $P = [a, b]$. Let $\Delta'(P)$ be the subcomplex of $\Delta(P)$ consisting of all nonmaximal chains. Since P is CM, $\Delta'(P)$ is also CM by Theorem 6.4. Restrict the map f defined above. This defines an order-preserving map $f\colon \Delta'(P)^* \to \operatorname{Int}'(P)^*$, where $\operatorname{Int}'(P) = \operatorname{Int}(P) \backslash \{[a, b]\}$, whose fibers are acyclic. By the Leray spectral sequence for f and the constant diagram,

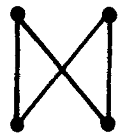
$$H^*(\Delta'(P), K) \cong H^*(\operatorname{Int}'(P), K).$$

Therefore $\operatorname{Int}'(P)$ is a bouquet. It follows that $\operatorname{Int}(P)$ is ACM.

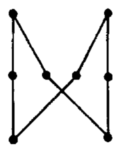
Conversely, suppose that $\operatorname{Int}(P)$ is ACM. Suppose that $x \in P$, and we consider the open interval $(\hat{0}, x)$ of $\hat{P}$. It need not be true that $\operatorname{Int}((\hat{0}, x))$ is an open subinterval of $\widehat{\operatorname{Int}(P)}$. However, let x be a maximal element of P. Then $V([x, x]) \cong J(x) \times V(x) \cong J(x)$ since $V(x) = \{x\}$. Now unless P consists of only one element, in which case everything is trivial, $V([x, x])$ is CM. Therefore, $J(x)$ is CM. Similarly, if x is a minimal element of P, then $V(x)$ is CM. It follows that P is ACM. ∎

We next consider two less familiar operations. If P and Q are posets, the *cardinal power* of P and Q, denoted either by P^Q or by $\operatorname{Hom}(Q, P)$, is the set of all order-preserving functions $f\colon Q \to P$, ordered componentwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in Q$. In other words, P^Q is given the induced order as a subset of the product $P \times P \times \cdots \times P$ of $|Q|$ copies of P. In the special case for which P is a two-element chain, we write 2^Q for the cardinal power P^Q. It is easy to see that 2^Q is isomorphic to the distributive lattice of order-filters of Q ordered by inclusion.

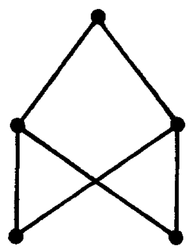
In general P^Q is not ACM even if both P and Q are CM. For example, let P be the poset



and let Q be the poset ; then P^Q is the poset

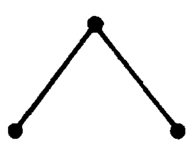


which is far from being ACM. As another example, use



for P and  for Q. Then P^Q is not ACM even though both P and Q are acyclic and CM.

As a final example let P be the poset



The cardinal power P^P is not even ranked even though P is an acyclic CM poset. Although it appears that we can say little about the cardinal power with regard to the CM property, it is possible that $\hat{P}^Q$ is CM if P is CM. This is suggested by the analogous property for lexicographic shellability, which was shown by Björner [10].

We now consider a rather less familiar operation. Let $f: P \rightarrow Q$ be a surjective order-preserving map such that $x \leqslant y$ in P if and only if $f(x) \leqslant f(y)$ in Q. We may think of P as being obtained by forming the (disjoint) union of the posets $f^{-1}(x)$ for $x \in Q$ and then decreeing that if $x \leqslant y$ in Q then every element of $f^{-1}(x)$ is to be below every element of $f^{-1}(y)$. This is quite a different method for "constructing" posets from the fibration method described in Section 5. In contrast to the fibration method, one rarely finds that P is ACM in the situation above even if Q and all the inverse images $f^{-1}(x)$ are CM. However, there is a special case that works. If the inverse images $f^{-1}(x)$ are all antichains we will say that P is obtained by *replicating* elements of Q.

THEOREM 7.3. *Let P be obtained by replicating elements of Q. Then P is* CM *or* ACM *if and only if Q is also.*

Proof. By the obvious induction we need only consider the special case for which P has one more element than Q, say that $f\colon P \to Q$ is the natural map and that $f^{-1}(x_0) = \{x_1, x_2\}$. Moreover, we may also assume by induction that P and Q are ACM.

Assume that Q is CM. We wish to show that P is a bouquet. If P has a minimum element there is nothing to prove. By Theorem 5.2 applied to the map $f\colon P \to Q$, we are reduced to the case for which x_0 is the minimum element of Q. We now take the order-dual of P and Q and proceed as before. As a result we may assume that x_0 is both the minimum and maximum element of Q. This case is trivial so we are done.

We now consider the converse. Assume that P is CM. We wish to show that Q is a bouquet. Consider the Leray spectral sequence for the map $\hat{f}\colon P\hat{1} \to Q\hat{1}$, where $\hat{f}$ is the extension of f such that $\hat{f}(\hat{1}) = \hat{1}$. Now if $x \neq x_0$ or $\hat{1}$, then $R^q\hat{f}_*K[\hat{1}]_x = 0$, while

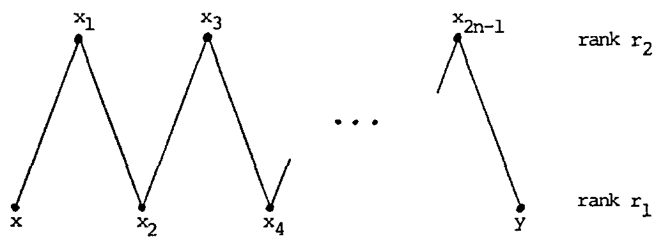
$$R^q\hat{f}_*K[\hat{1}]_{\hat{1}} = \begin{Bmatrix} K & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{Bmatrix}$$

and $(R^q\hat{f}_*K[\hat{1}])_{x_0} = H^q(\hat{f}/x_0, K[\hat{1}]) = \tilde{H}^{q-1}(f/x_0, K)$. By the first part of our proof, f/x_0 is CM. Therefore $(R^q\hat{f}_*K[\hat{1}])_{x_0} = 0$ for $q \neq r(f/x_0)$. Therefore $R^0\hat{f}_*K[\hat{1}] = K[\hat{1}]$ and $R^{r(f/x_0)}\hat{f}_*K[\hat{1}]$ are the only nonzero direct images of $K[\hat{1}]$. We know that $H^p(Q\hat{1}, K[\hat{1}]) \cong \tilde{H}^{p-1}(Q, K)$, but we can say nothing about it yet. The only other terms in the spectral sequence are $H^p(Q\hat{1}, R^{r(f/x_0)}\hat{f}_*K[\hat{1}])$. Since $R^{r(f/x_0)}\hat{f}_*K[\hat{1}]$ is supported on x_0, the cohomology vanishes except for $p = r(x_0) - 1$. Since $r(x_0) + r(f/x_0) = r(P) + 1$, this term contributes only to $H^{r(P)}(P\hat{1}, K[\hat{1}])$ in the abutment of the spectral sequence. Therefore the Leray spectral sequence tells us that $H^n(P\hat{1}, K[\hat{1}]) = H^n(Q\hat{1}, K[\hat{1}])$ for $n < r(P) = r(Q)$. Since P is CM, it follows that Q is also. ∎

8. COMBINATORIAL HOMOTOPY

In this last section we develop a combinatorial concept of homotopy that is applicable to arbitrary CM posets. Although we were originally motivated by the work of Tutte [38], our result is more closely related to the Hurewicz theorem relating homotopy groups with homology groups (see Spanier [30, Section 7.5]). In contrast to the topological theory, we can offer no satisfactory concept of "higher" homotopy groups. We add that the Homotopy Theorem (8.1), turns out to be quite useful for showing that particular examples of posets are CM. See [6, Example 2.10].

Combining the rank selection theorem with the characterization of CM posets of length 1, we see that if r_1 and r_2 are any two ranks of a CM poset P and if $x, y \in P$ both have rank r_1, then there is a sequence $x = x_0, x_1, \ldots, x_{2n} = y$ of elements of P such that the x_i's are alternately of ranks r_1 and r_2 and such that x_i and x_{i+1} are comparable for all i. We call such a sequence a *path* from x to y.



More precisely, for a ranked poset P we define a *path along ranks* $r_1 < r_2$ to be a function

$f\colon \{0, \ldots, 2n\} \to P$ such that

$$r(f(m)) = \begin{cases} r_1 & \text{if } m \text{ is even} \\ r_2 & \text{if } m \text{ is odd} \end{cases}$$

and such that for all odd m, $f(m) \geqslant f(m-1), f(m+1)$. We call n the *length* of the path, and we call $f(0)$ and $f(2n)$ the *endpoints* of the path. A path is a *loop* at x if $f(0) = f(2n) = x$.

Given a notion of "path" one immediately has a concept of a fundamental "groupoid." Choosing a reasonable notion of "homotopy" of paths, one obtains a notion of a fundamental group. More precisely, let f and g be paths in a ranked poset along ranks $r_1 < r_2$. If the final endpoint of f is the initial endpoint of g, then we may speak of the *product* $h = fg$ of the paths f and g given by:

$$h(j) = \begin{cases} f(j) & \text{if } 0 \leqslant j \leqslant 2n, \\ g(j-2n) & \text{if } 2n < j \leqslant 2n + 2m, \end{cases}$$

where n and m are the lengths of f and g respectively.

To define a notion of homotopy of paths we must specify three ranks $r_1 < r_2 < r_3$. Let f and g be two paths along ranks $r_1 < r_2$ having the same endpoints. We say f and g are *simply homotopic in rank* r_3 if there are paths h_1, h_2, f', g' along ranks $r_1 < r_2$ and an element $c \in P$ of rank r_3 such that

(1) $f = h_1 f' h_2$ and $g = h_1 g' h_2$,

(2) $c \geqslant f'(j)$ and $c \geqslant g'(k)$ for all j, k.

We call c the *center* of the *simple homotopy*. We will say that f and g are *combinatorially homotopic in rank* r_3 if there is a sequence of simple homotopies in rank r_3 joining f to g.

Fix an element x of rank r_1. The combinatorial homotopy classes in rank r_3 of loops at x along ranks $r_1 < r_2$ clearly define a group under the operation of product of paths. We write $\pi(P_S, x)$ for this group, where $S = \{r_1, r_2, r_3\}$. If P_S is connected and if x' is another element of rank r_1, then there is a (non-unique) isomorphism of $\pi(P_S, x')$ with $\pi(P_S, x)$. We follow Spanier [30, Section 7.4] in writing $\pi'(P_S, x)$ for $\pi(P_S, x)$ modulo its commutator subgroup.

HOMOTOPY THEOREM 8.1. *Let P be a connected poset of rank n which is* ACM *over $\mathbb{Z}$. Let $S = \{r_1 < r_2 < r_3\} \subseteq [n]$ be a set of three ranks of P and let $x \in P$ be of rank r_1. Then there is a group isomorphism $\pi'(P_S, x) \cong H_1(P, \mathbb{Z})$.*

Proof. The homology version of Theorem 6.5 is true for an arbitrary principal ideal domain. Thus $H_1(P, \mathbb{Z}) \cong H_1(P_S, \mathbb{Z})$. We also note that $H_0(P, \mathbb{Z}) \cong H_0(P_S, \mathbb{Z})$ so P_S is also connected. Therefore we may assume without loss of generality that $P = P_S$.

Now the elements of $\Delta(P)$ have a natural orientation given by the ordering on P. We find it convenient to employ the following notation. If $a_0, \ldots, a_l$ are elements of P, we define $(a_0; a_2; \ldots; a_l)$ to be 0 if either two of the a_i's are equal or $\{a_0, \ldots, a_l\}$ is not in $\Delta(P)$, and otherwise to be $\pm(a_{i_0} < \cdots < a_{i_l}) \in C_l(\Delta(P), \mathbb{Z})$, where the sign is chosen to be the sign of the permutation needed to put $\{a_0, \ldots, a_l\}$ in order.

Now for any path f in P we define $[f]$ to be the sum

$$[f] = \sum_{0 \leqslant i < 2n} (f(i); f(i+1)) \in C_2(\Delta(P), \mathbb{Z}).$$

It is easy to verify that if f and g are combinatorially homotopic then $[f]$ is homologous to $[g]$, and if f and g are two loops at x then $[fg] = [f] + [g]$. Moreover, if f is a loop, then $[f]$ is a 1-cycle. Therefore $f \mapsto [f]$ defines a group homomorphism $\pi'(P, x) \to H_1(P, \mathbb{Z})$, since $H_1(P, \mathbb{Z})$ is commutative.

We first show surjectivity. It is easy to see that every 1-cycle in P can be written in the form $\sum_{i=1}^{2N} (a_{i-1}; a_i)$, where for all i, $(a_{i-1}; a_i) \neq 0$, and where $a_0 = a_{2N}$. We show that the 1-cycle $\sum_{i=1}^{2N} (a_{i-1}; a_i)$ is homologous to a 1-cycle of the same form for which all the a_i's have ranks either 1 or 2. We do this in two steps.

We first prove that the 1-cycle is homologous to one whose a_i's are alternately minimal and maximal elements. We do this by induction as follows. Suppose that a_i is neither minimal nor maximal.

Case 1. $a_{i-1} > a_i > a_{i+1}$. Choose a minimal element $a_i' < a_i$. Then $(a_{i-1}\,;\,a_i) + (a_i\,;\,a_{i+1})$ is homologous to $(a_{i-1}\,;\,a_i') + (a_i'\,;\,a_{i+1})$.

Case 2. $a_{i-1} < a_i > a_{i+1}$. Choose a maximal element $a_i' > a_i$ and proceed as in Case 1.

Case 3. $a_{i-1} < a_i < a_{i+1}$. Then $(a_{i-1}\,;\,a_i) + (a_i\,;\,a_{i+1})$ is homologous to $(a_{i-1}\,;\,a_{i+1})$.

Case 4. $a_{i-1} > a_i > a_{i+1}$. Proceed as in Case 3.

In every case above either a_i is eliminated or a_i is replaced by a minimal or by a maximal element. Therefore we may assume that a_i is maximal for all even subscripts.

Now for each even subscript we define a path f_i as follows. Since P is ACM, we know that $J(a_i)$ is CM for all even i. Therefore there is a path along ranks 1 and 2 from a_{i-1} to a_{i+1}. Let f_i be one such path. Then $(a_{i-1}\,;\,a_i) + (a_i\,;\,a_{i+1})$ is homologous to $[f_i]$; indeed, if f_i has length n, then the boundary of

$$\sum_{j=0}^{2n-1} (f_i(j); f_i(j+1); a_i)$$

is $[f_i] - (a_{i-1}\,;\,a_i) + (a_{i+1}\,;\,a_i)$.

Therefore $\sum_{i=1}^{2N} (a_{i-1}\,;\,a_i)$ is homologous to $\sum_{i=1}^{N} [f_{2i}] = [\prod_{i=1}^{N} f_{2i}]$. Set $f = \prod_{i=1}^{N} f_{2i}$. Then f is a loop at a_0. Since P is connected, $P_{\{1,2\}}$ is also connected by Theorem 6.5. So there is a path g along ranks 1 and 2 from a_0 to x. Now $g^{-1}fg$ is a loop at x and $[g^{-1}fg]$ is homologous to $\sum_{i=1}^{2N} (a_{i-1}\,;\,a_i)$. Surjectivity then follows.

We now show that the map $\pi'(P, x) \to H_1(P, \mathbb{Z})$ is an isomorphism. Let f be a loop at x such that $[f]$ is homologous to zero. Then $[f]$ is the boundary of a 2-chain $w = \sum_{i=1}^{N} (a_i\,;\,b_i\,;\,c_i)$. We can write w so that none of the terms $(a_i\,;\,b_i\,;\,c_i)$ vanish, so that c_i is always larger than a_i and b_i and so that no terms can cancel any other. Clearly, up to order of summands, there is a unique way to write w as specified above.

We now collect together the terms of w having the same largest element. This enables us to rearrange w as a double sum;

$$w = \sum_{c \in P_3} \sum_{j=1}^{M_c} (a_j(c);\, b_j(c);\, c).$$

Now f was assumed to be a path along ranks 1 and 2. Therefore it contains no maximal elements of P. Therefore, each 1-chain $\sum_{j=1}^{M_c} (a_j(c); b_j(c))$ is actually a 1-cycle, i.e. for each $c \in P_3$, there is a loop f_c such that $\sum_{j=1}^{M_c} (a_j(c); b_j(c)) = [f_c]$. Moreover, since the boundary of w is $[f]$, we conclude that $[f] = \sum_{c \in P_3} [f_c]$.

Assume that $[f] \neq 0$. Then there exists an element d of rank 3 such that $[f]$ and $[f_d]$ have a 1-chain $(a < b)$ in common. Without loss of generality we may suppose that $(f(j-1) < f(j)) = (f_d(0) < f_d(1))$. Let h_1 be the restriction of f to $\{0,..., j-1\}$ and define h_2 so that $f = h_1 h_2$. Then f is homotopic to $h_1 f_d^{-1} h_2$ with center d. Now $[h_1 f_d^{-1} h_2] = [h_1 h_2] - [f_d] = [f] - [f_d] = \sum_{c \in P_3 \setminus \{d\}} [f_c]$. Continuing this procedure successively, we eventually find that f is homotopic to some loop f' at x such that $[f'] = 0$.

Thus we may assume that $[f] = 0$. Let n be the length of f. This means that there is a permutation σ of the odd integers between 0 and $2n$ such that for all odd j,

$$(f(\sigma(j)-1) < f(\sigma(j))) = (f(j+1) < f(j)).$$

As noted earlier, $P_{\{1,2\}}$ is connected. Therefore for every $a \in P$ we may choose a path $g[a]$ from x to a. In the special case $a = x$, we take $g[x]$ to be the trivial loop at x. Write f as a product of n paths each of length 1: $f = h[1]\, h[3] \cdots h[2n-1]$. Then f is trivially homotopic to $g[f(0)]\, h[1]\, g[f(2)]^{-1} g[f(2)]\, h[3] \cdots h[2n-1]\, g[f(2n)] = \prod_{j \text{ odd}} g[f(j-1)]\, h[j]\, g[f(j+1)]^{-1}$. Now each of the factors $g[f(j-1)]\, h[j]\, g[f(j+1)]^{-1}$ is a loop at x. So in $\pi'(P, x)$ we may rearrange them as we please. In particular, we may arrange them according to the orbits of σ.

Let $0 < j < 2n$ be odd and suppose that $\sigma^m(j) = j$ but that $\sigma^k(j) \neq j$ for $0 < k < m$. Then

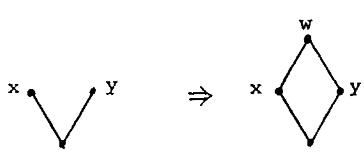
$$\prod_{k=0}^{m-1} g[f(\sigma^k(j)-1)]\, h[\sigma^k(j)] g[f(\sigma^k(j)+1)]^{-1}$$

is homotopic to

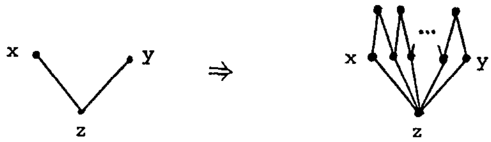
$$g[f(j-1)] \left(\prod_{k=0}^{m-1} h[\sigma^k(j)] \right) g[f(j-1)]^{-1} \tag{*}$$

because $f(\sigma^k(j)+1) = f(\sigma^{k+1}(j)-1)$ for all k. Now all the paths $h[\sigma^k(j)]$ have the *same* top element $f(j) = f(\sigma(j)) = \cdots$. Therefore $(*)$ is homotopic to the trivial loop at x. Hence f is equivalent to the trivial loop in $\pi'(P, x)$. ∎

We may interpret the Homotopy Theorem as implying that CM posets obey a weak semimodularity law. More precisely, the ordinary semimodular law says that if x, y cover z then there is a w covering x and y:



For CM posets, if x and y cover z, then there is only a path of coverings from x to y:



but these paths are all "combinatorially homotopic" (at least in the sense of $\pi'(P, x)$ rather than $\pi(P, x)$).

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