HW1 grades are out

HW2 due tonight

HW3 out by Friday, due 10/5

CS3000: Algorithms & Data
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Lecture 6:
- Dynamic Programming:
  Fibonacci Numbers, Interval Scheduling

Sep 25, 2018

Dynamic Programming

- Don't think too hard about the name
  - *I thought dynamic programming was a good name. It was something not even a congressman could object to. So I used it as an umbrella for my activities.* -Bellman

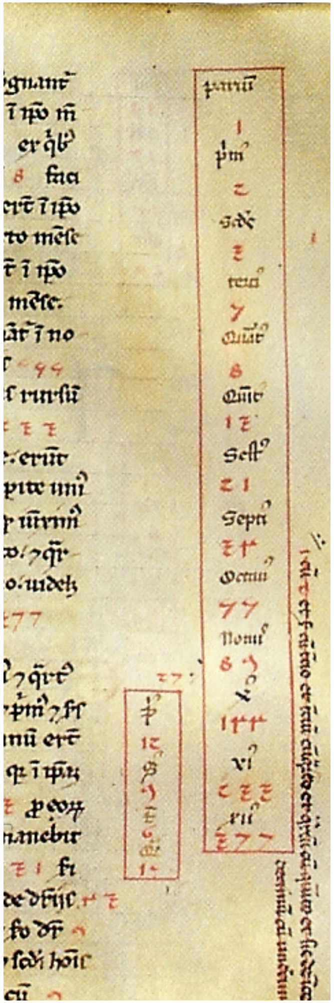
- Dynamic programming is careful recursion
  - Break the problem up into small pieces
  - Recursively solve the smaller pieces
  - **Key Challenge:** identifying the pieces

Warmup: Fibonacci Numbers

Fibonacci Numbers

- $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- $F(n) = F(n-1) + F(n-2)$

- $F(n) \to \phi^n \approx 1.62^n$
- $\phi = \left(\frac{1+\sqrt{5}}{2}\right)$ is the golden ratio



Fibonacci Numbers: Take I

```
FibI(n):
  If (n = 0): return 0
  ElseIf (n = 1): return 1
  Else: return FibI(n-1) + FibI(n-2)
```

- How many recursive calls does **FibI(n)** make?
  - 2^n

$T(n) = \#$ of recursive calls made by Fib(n)

$T(n) = T(n-1) + T(n-2)$

$T(n) \approx \phi^n \approx 1.62^n$

Fibonacci Numbers: Take II

"Memoization" "Top-Down"

```
M ← empty array, M[0]←0, M[1]←1
FibII(n):
  If (M[n] is not empty): return M[n]
  ElseIf (M[n] is empty):
    M[n] ← FibII(n-1) + FibII(n-2)
    return M[n]
```

- How many recursive calls does **FibII(n)** make?
  - Only have to fill n-1 entries
  - Each pair of recursive calls fills one entry

$\Rightarrow$ • 2n-2 recursive calls $O(n)$

Fibonacci Numbers: Take III

"Bottom-Up"

```
FibIII(n):
  M[0] ← 0, M[1] ← 1
  For i = 2,…,n:
    M[i] ← M[i-1] + M[i-2]
  return M[n]
```

- What is the running time of **`FibIII(n)`**?

M[0]	M[1]	M[2]	3	4	5	6	7
0	1	1	2	3	5	8	13

$O(n^2)$ time algorithm (b/c Fib(n) has $\Omega(n)$ digits)

$\text{Fib}(n) \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$

Fibonacci Numbers

- $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- $F(n) = F(n-1) + F(n-2)$

- Solving the recurrence recursively takes $\approx 1.62^n$ time
  - Problem: Recompute the same values $F(i)$ many times
- Two ways to improve the running time
  - Remember values you've already computed ("top down")
  - Iterate over all values $F(i)$ ("bottom up")

- **Fact:** Can solve even faster using Karatsuba's algorithm!

Dynamic Programming:
Interval Scheduling

Interval Scheduling

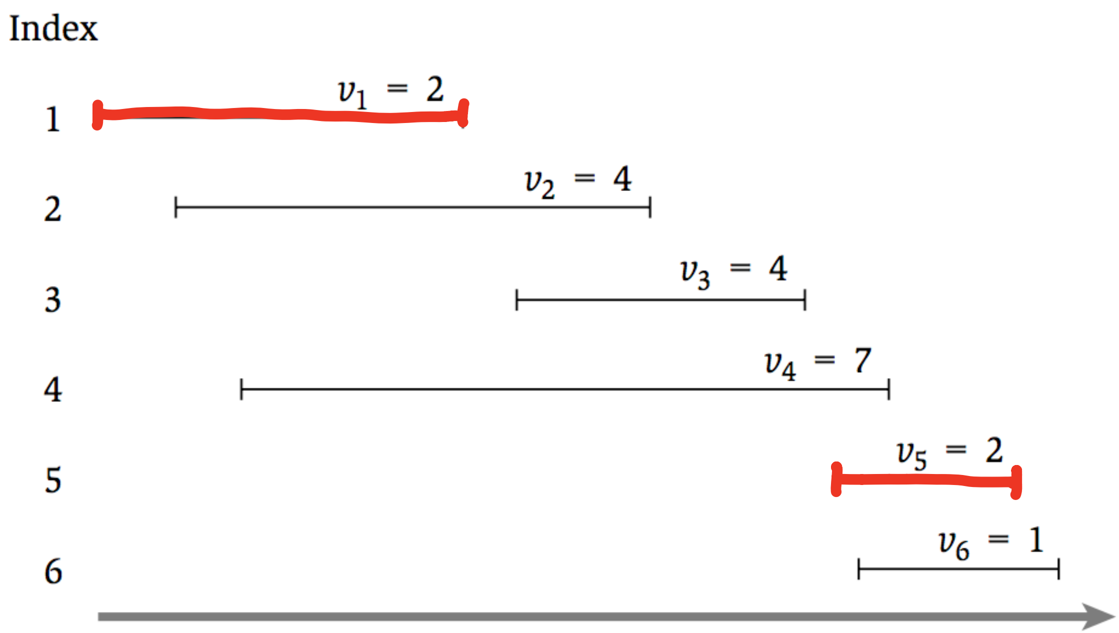
- How can we optimally schedule a resource?
  - This classroom, a computing cluster, …

- **Input:** n intervals (s_i, f_i) each with value v_i
  - Assume intervals are sorted so $f_1 < f_2 < \cdots < f_n$
- **Output:** a compatible schedule S maximizing the total value of all intervals
  - A **schedule** is a subset of intervals $S \subseteq \{1, \ldots, n\}$
  - A schedule S is **compatible** if no $i, j \in S$ overlap
  - The **total value** of S is $\sum_{i \in S} v_i$

Interval Scheduling

$S = \{1, 5\}$ value$(S) = v_1 + v_5$

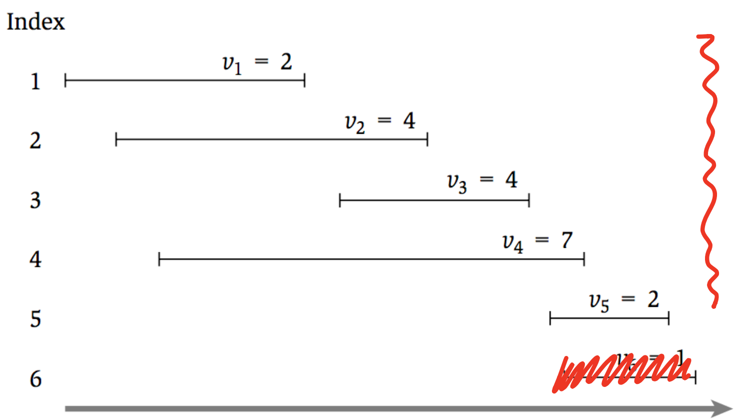
$= 4$



A Recursive Formulation

- Let O be the **optimal** schedule
- **Case 1:** Final interval is not in O (i.e. $6 \notin O$)
  - Then O must be the optimal solution for $\{1, \ldots, 5\}$

If O were not the optimal of $\{1, \ldots, 5\}$ and $6 \notin O$, then the opt of $\{1, \ldots, 5\}$ is better than O.



O is the opt of these intervals

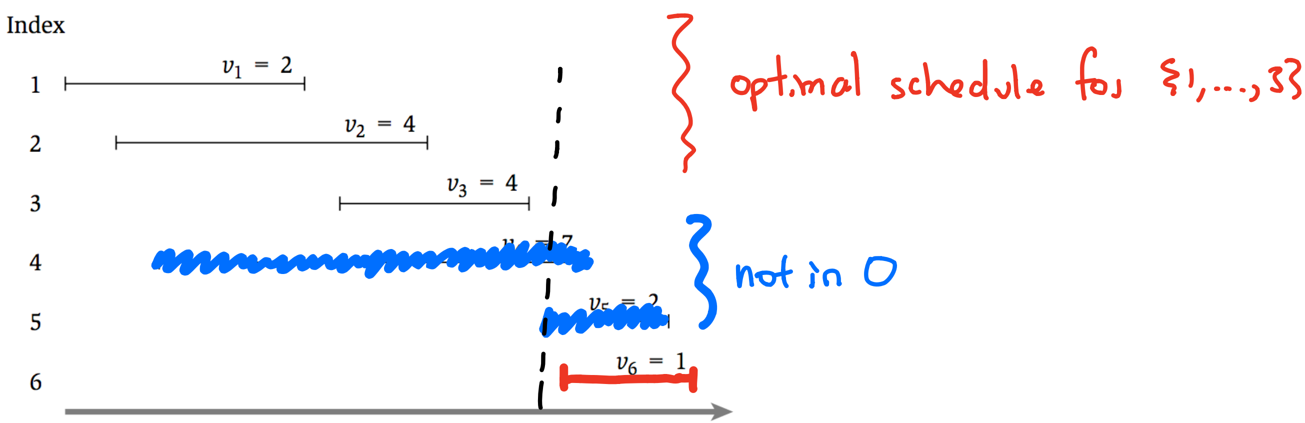
$6 \notin O$

A Recursive Formulation

- Let O be the **optimal** schedule
- **Case 2:** Final interval is in O (i.e. $6 \in O$)
  - Then O must be $\{6\}$ + the optimal solution for $\{1, \ldots, 3\}$

O is either $\{6\} + \mathrm{opt}(\{1,2,3\})$

$\mathrm{opt}(\{1,\ldots,5\})$



A Recursive Formulation

O_n is the thing we want

- Let O_i be the **optimal schedule** using only the intervals $\{1, \ldots, i\}$
- **Case 1:** Final interval is not in O_i ($i \notin O_i$) $O_i = O_{i-1}$
  - Then O_i must be the optimal solution for $\{1, \ldots, i-1\}$
- **Case 2:** Final interval is in O ($i \in O_i$) $O_i = \{i\} + O_{p(i)}$
  - Assume intervals are sorted so that $f_1 < f_2 < \cdots < f_n$
  - Let $p(i)$ be the largest j such that $f_j < s_i$
  - Then O_i must be $\{i\}$ + the optimal solution for $\{1, \ldots, p(i)\}$

If value(O_{i-1}) > v_i + value$(O_{p(i)})$ then i is <u>not</u> in O_i

Else i <u>is</u> in O_i

A Recursive Formulation

OPT(i) = value(O_i)

- Let $OPT(i)$ be the **value of the optimal schedule** using only the intervals $\{1, \ldots, i\}$
- **Case 1:** Final interval is not in O_i ($i \notin O_i$) OPT(i) = OPT(i-1)
  - Then O must be the optimal solution for $\{1, \ldots, i-1\}$
- **Case 2:** Final interval is in O_i ($i \in O_i$) OPT(i) = v_i + OPT(p(i))
  - Assume intervals are sorted so that $f_1 < f_2 < \cdots < f_n$
  - Let $p(i)$ be the largest j such that $f_j < s_i$
  - Then O must be $\{i\}$ + the optimal solution for $\{1, \ldots, p(i)\}$

- $OPT(i) = \max\{OPT(i-1), v_i + OPT(p(i))\}$
- $OPT(0) = 0, OPT(1) = v_1$

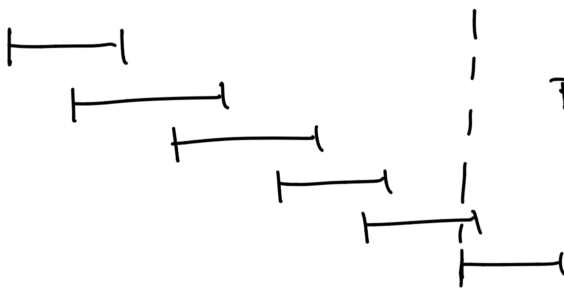
Algorithmically the same as computing Fib #s

Interval Scheduling: Take I

Assuming values p(n) are already computed

```
// All inputs are global vars
FindOPT(n):
  if (n = 0): return 0
  elseif (n = 1): return v₁
  else:
    return max{FindOPT(n-1), vₙ + FindOPT(p(n))}
```

- What is the running time of **FindOPT(n)**?



$p(i) = i - 2$

At least 1.62^n recursive calls

Interval Scheduling: Take II

```
// All inputs are global vars
M ← empty array, M[0] ← 0, M[1] ← v_1
FindOPT(n):
  if (M[n] is not empty): return M[n]
  else:
    M[n] ← max{FindOPT(n-1), v_n + FindOPT(p(n))}
    return M[n]
```

- What is the running time of **`FindOPT(n)`**?

(n-1 entries to fill) × (2 calls per entry) = 2n-2

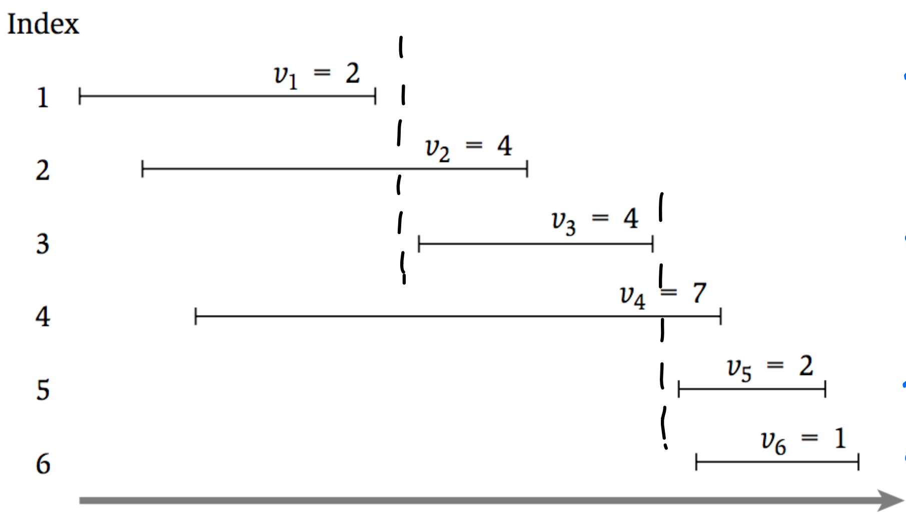
$O(n)$ time (+ $O(n \log n)$ to sort if necessary
+ $O(n)$ to compute $p(1) \dots p(n)$)

Interval Scheduling: Take II

$M[4] = \max\{M[3],\ 7 + M[0]\}$

$M[5] = \max\{M[4],\ 2 + M[3]\}$

$M[6] = \max\{M[5],\ 1 + M[3]\}$



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

$M[i] = OPT(i)$

$M[2] = \max\{M[1],\ 4 + M[0]\}$

$M[3] = \max\{M[2],\ 4 + M[1]\}$

M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]
0	2	4	6	7	8	8

Interval Scheduling: Take III

"Bottom-Up Dynamic Programming"

// All inputs are global vars

FindOPT(n):

  M[0] ← 0, M[1] ← ~~1~~ v_1

  for (i = 2,…,n): $\max\{M[i-1], v_i + M[p(i)]\}$

    M[i] ← ~~[illegible]~~

  return M[n]

- What is the running time of **FindOPT(n)**?

$O(n)$ + time to sort if needed + time to compute $p(i)$'s

Finding the Optimal Solution

- Let $OPT(i)$ be the **value of the optimal schedule** using only the intervals $\{1, \ldots, i\}$
- **Case 1:** Final interval is not in O ($i \notin O$)
- **Case 2:** Final interval is in O ($i \in O$)

- $OPT(i) = \max\{OPT(i-1), v_i + OPT(p(i))\}$

If $(OPT(i-1) > v_i + OPT(p(i)))$:

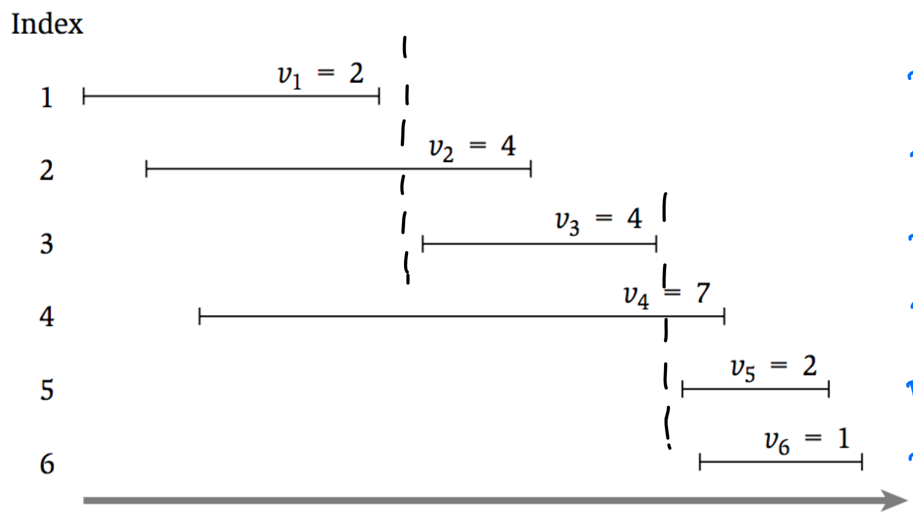
$O_i = O_{i-1}$ $(i \notin O_i)$

Else If $(v_i + OPT(p(i)) > OPT(i-1))$:

$O_i = \{i\} + O_{p(i)}$ $(i \in O_i)$

Else: O_i could be either $\{i\} + O_{p(i)}$ or O_{i-1}

Interval Scheduling: Take II



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

$M[i] = OPT(i)$

$O_6 = \{1, 3, 5\}$

M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]
0	2	4	6	7	8	8

$O_1 = \{1\}$ $\quad 3 \in O \quad 5 \in O \quad 6 \notin O$

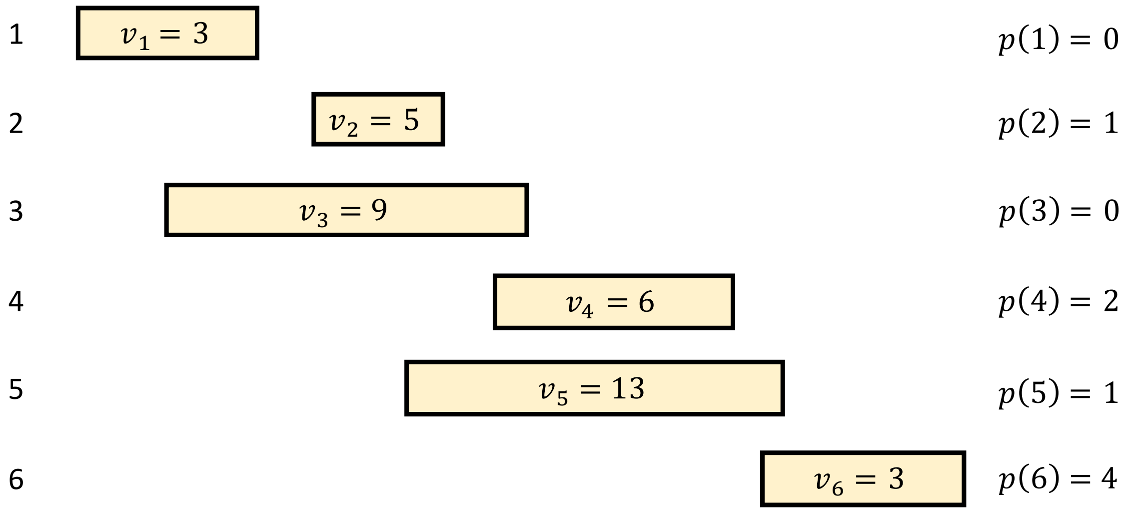
Interval Scheduling: Take III

Completed table with value of optimum

```
// All inputs are global vars
FindSched(M,n):
  if (n = 0): return ∅
  elseif (n = 1): return {1}
  elseif (vₙ + M[p(n)] > M[n-1]):
    return {n} + FindSched(M,p(n))
  else:
    return FindSched(M,n-1)
```

- What is the running time of **`FindSched(n)`**?

Now You Try



M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]

Dynamic Programming Recap

- Express the optimal solution as a **recurrence**
  - Identify a small number of subproblems
  - Relate the optimal solution on subproblems
- Efficiently solve for the **value** of the optimum
  - Simple implementation is exponential time
  - Top-Down: store solution to subproblems
  - Bottom-Up: iterate through subproblems in order
- Find the **solution** using the table of **values**