• HW2 due tonight

• HW1 grades returned

solutions on Piazza

82 mean

88 median

• HW3 out by Friday

due next Friday

CS3000: Algorithms & Data
Jonathan Ullman

Lecture 6:

- Dynamic Programming: Fibonacci Numbers, Interval Scheduling

Sep 25, 2018

Dynamic Programming

- Don't think too hard about the name
  - *I thought dynamic programming was a good name. It was something not even a congressman could object to. So I used it as an umbrella for my activities.* -Bellman

- Dynamic programming is careful recursion
  - Break the problem up into small pieces
  - Recursively solve the smaller pieces
  - **Key Challenge:** identifying the pieces

Divide and Conquer : speeding up simple algorithms

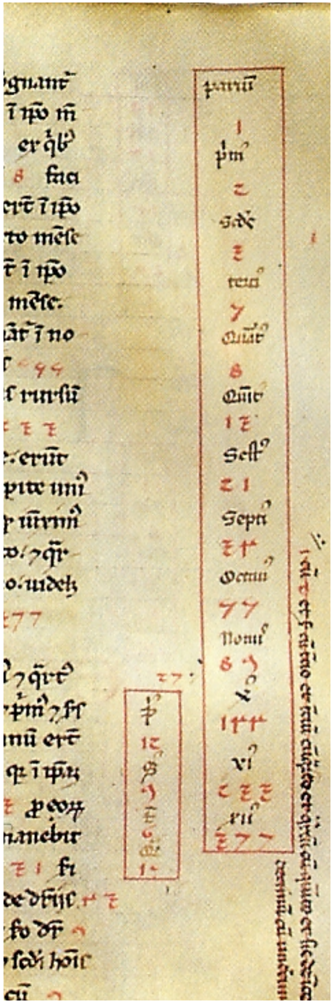
Dynamic Programming : often the only polynomial time alg

Warmup: Fibonacci Numbers

Fibonacci Numbers

- $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- $F(n) = F(n-1) + F(n-2)$

- $F(n) \to \phi^n \approx 1.62^n$
- $\phi = \left(\frac{1+\sqrt{5}}{2}\right)$ is the golden ratio



Fibonacci Numbers: Take I

```
FibI(n):
  If (n = 0): return 0
  ElseIf (n = 1): return 1
  Else: return FibI(n-1) + FibI(n-2)
```

- How many recursive calls does **FibI(n)** make?

  - 2^n
  - $2n$

$T(n) = \#$ of calls made by FibI(n)

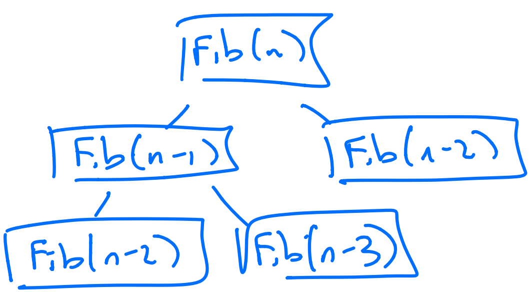
$T(n) = T(n-1) + T(n-2)$

$T(0) = 0$

$T(1) = 0$

$T(2) = 2$

$T(n) = F(n) \approx 1.62^n$



Fibonacci Numbers: Take II

"Memoization", "Top-Down"

```
M ← empty array, M[0] ← 0, M[1] ← 1
FibII(n):
  If (M[n] is not empty): return M[n]
  ElseIf (M[n] is empty):
    M[n] ← FibII(n-1) + FibII(n-2)
    return M[n]
```

- How many recursive calls does **`FibII(n)`** make?

Array has $n+1$ elements, need to fill $n-1$

Each time we make a pair of recursive calls, we fill one $M[i]$

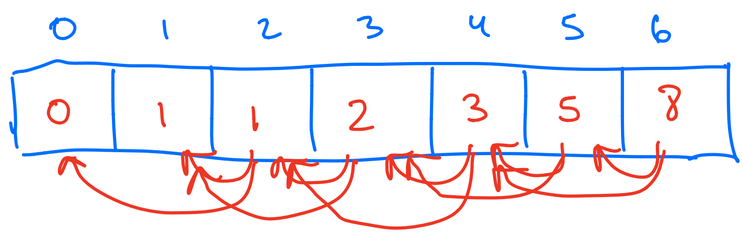
$\Rightarrow \leq 2(n-1) = O(n)$ recursive calls

Fibonacci Numbers: Take III

"Bottom-Up"

```
FibIII(n):
  M[0] ← 0, M[1] ← 1
  For i = 2,…,n:
    M[i] ← M[i-1] + M[i-2]
  return M[n]
```

- What is the running time of **FibIII(n)**?



$$F(n) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

n-1 additions, each addition involves $\Theta(n)$-digit numbers

$\Rightarrow \Theta(n^2)$ time

Fibonacci Numbers

- $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- $F(n) = F(n-1) + F(n-2)$

- Solving the recurrence recursively takes $\approx 1.62^n$ time
  - Problem: Recompute the same values $F(i)$ many times
- Two ways to improve the running time
  - Remember values you've already computed ("top down")
  - Iterate over all values $F(i)$ ("bottom up")

- **Fact:** Can solve even faster using Karatsuba's algorithm!

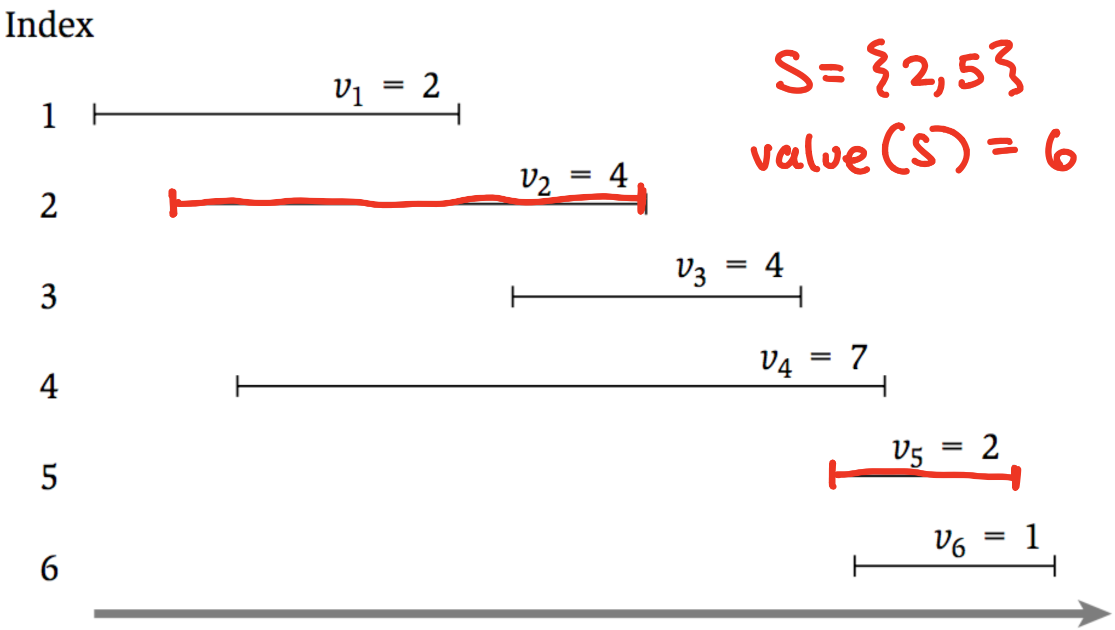
Dynamic Programming: Interval Scheduling

Interval Scheduling (Weighted)

- How can we optimally schedule a resource?
  - This classroom, a computing cluster, …

- **Input:** n intervals (s_i, f_i) each with value v_i
  - Assume intervals are sorted so $f_1 < f_2 < \cdots < f_n$
- **Output:** a compatible schedule S maximizing the total value of all intervals
  - A **schedule** is a subset of intervals $S \subseteq \{1, \ldots, n\}$
  - A schedule S is **compatible** if no $i, j \in S$ overlap
  - The **total value** of S is $\sum_{i \in S} v_i$

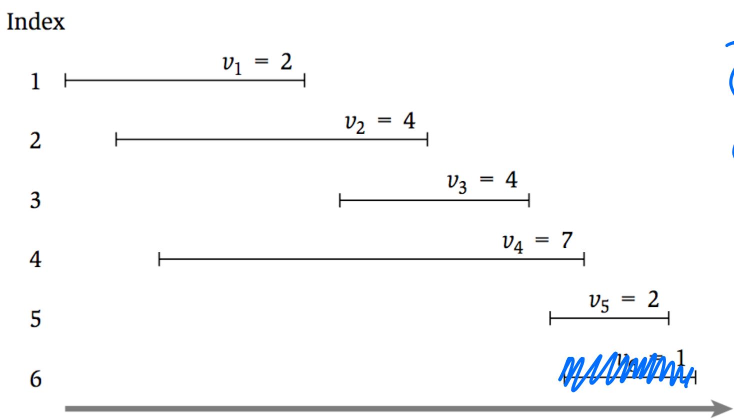
Interval Scheduling



A Recursive Formulation

- Let O be the **optimal** schedule
- **Case 1:** Final interval is not in O (i.e. $6 \notin O$)
  - Then O must be the optimal solution for $\{1, \dots, 5\}$

If O were <u>not</u> the optimal sched for $\{1, \dots, 5\}$
then O is <u>not</u> the optimal sched for $\{1, \dots, 6\}$

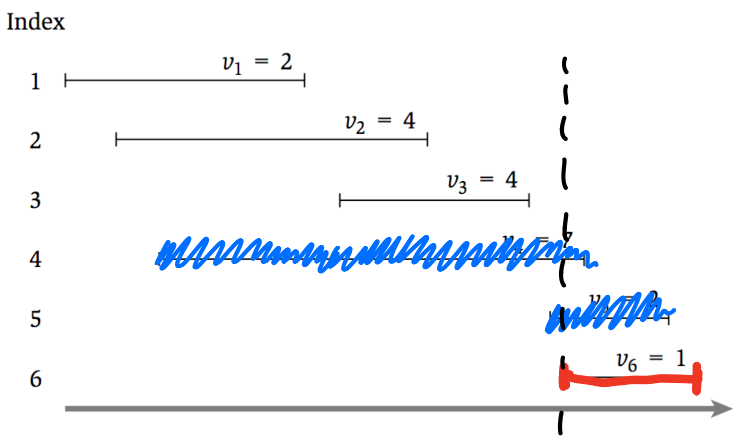


O is opt on these

A Recursive Formulation

- Let O be the **optimal** schedule
- **Case 2:** Final interval is in O (i.e. $6 \in O$)
  - Then O must be 6 + the optimal solution for $\{1, \dots, 3\}$

If $O \setminus \{6\}$ were not opt for $\{1,\dots,3\}$ then $\{6\}$ + [opt for $\{1,\dots,3\}$] is better than O



which is better?

① opt sched for $\{1,\dots,5\}$

② opt sched for $\{1,\dots,3\}$ + $\{6\}$

A Recursive Formulation

$n+1$ "subproblems"

- Let O_i be the **optimal schedule** using only the intervals $\{1, \dots, i\}$
- **Case 1:** Final interval is not in O $(i \notin O_i)$
  - Then O must be the optimal solution for $\{1, \dots, i-1\}$ (O_{i-1})
- **Case 2:** Final interval is in O $(i \in O_i)$
  - Assume intervals are sorted so that $f_1 < f_2 < \cdots < f_n$
  - Let $p(i)$ be the largest j such that $f_j < s_i$
  - Then O_i must be i + the optimal solution for $\{1, \dots, p(i)\}$

$$O_i = \{i\} + O_{p(i)}$$

A Recursive Formulation

a number, not a set

- Let $\boxed{OPT(i)}$ be the **value of the optimal schedule** using only the intervals $\{1, \ldots, i\}$
- **Case 1:** Final interval is not in O $(i \notin O_i)$
  - Then O must be the optimal solution for $\{1, \ldots, i-1\}$ O_{i-1}
- **Case 2:** Final interval is in O $(i \in O_i)$
  - Assume intervals are sorted so that $f_1 < f_2 < \cdots < f_n$
  - Let $p(i)$ be the largest j such that $f_j < s_i$
  - Then O must be i + the optimal solution for $\{1, \ldots, p(i)\}$

$\{i\} + O_{p(i)}$

- $OPT(i) = \max\{OPT(i-1), v_i + OPT(p(i))\}$
- $OPT(0) = 0, OPT(1) = v_1$

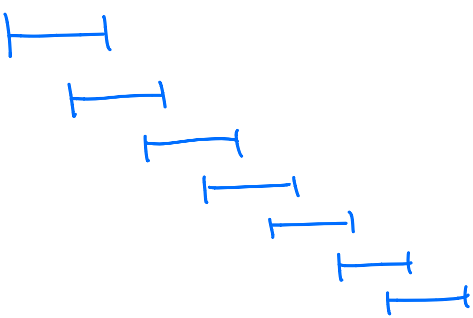
Interval Scheduling: Take I

assume $p(1), \dots, p(n)$ are computed

```
// All inputs are global vars
FindOPT(n):
  if (n = 0): return 0
  elseif (n = 1): return v_1
  else:
    return max{FindOPT(n-1), v_n + FindOPT(p(n))}
```

FindOPT(n-1), FindOPT(n-2)

- What is the running time of **`FindOPT(n)`**?



As many as 1.62^n recursive calls

$\forall i \quad p(i) = i-2$

Interval Scheduling: Take II

```
// All inputs are global vars
M ← empty array, M[0] ← 0, M[1] ← v₁
FindOPT(n):
  if (M[n] is not empty): return M[n]
  else:
    M[n] ← max{FindOPT(n-1), vₙ + FindOPT(p(n))}
    return M[n]
```

- What is the running time of **FindOPT(n)**?

Need to fill $\leq n-1$ entries of M

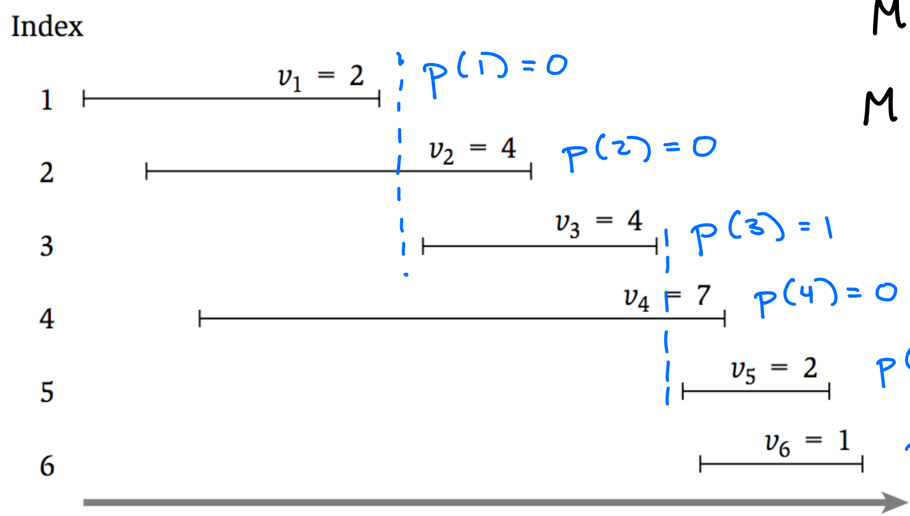
$\times$ 2 recursive calls / entry

$\leq 2(n-1)$ recursive calls

$O(n)$ running time

$+ O(n \log n)$ to sort by f_i

Interval Scheduling: Take II



$M[4] = \max\{6, 7 + 0\}$

$M[5] = \max\{7, 2 + 6\}$

$M[6] = \max\{8, 1 + 6\}$

$M[i] = OPT(i)$

$M[2] = \max\{M[1], 4 + M[0]\}$

$M[3] = \max\{M[2], 4 + M[1]\}$

M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]
0	2	4	6	7	8	8

Interval Scheduling: Take III

```
// All inputs are global vars
FindOPT(n):
  M[0] ← 0, M[1] ← v1
  for (i = 2,…,n):
    M[i] ← max{M[i-1], vi + M[p(i)]}
  return M[n]
```

- What is the running time of **`FindOPT(n)`**?

$O(n)$ (+ $O(n \log n)$ to sort if needed)

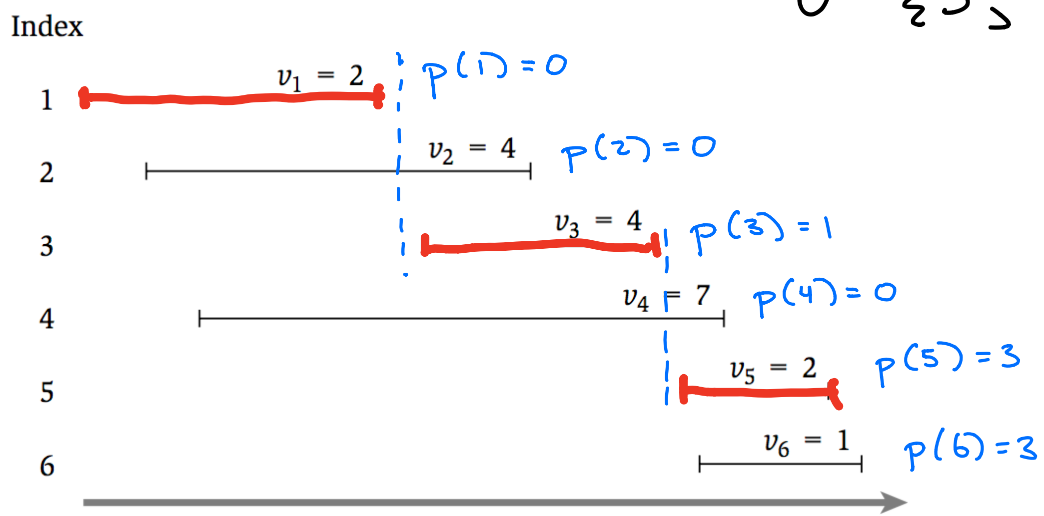
Finding the Optimal Solution

- Let $OPT(i)$ be the **value of the optimal schedule** using only the intervals $\{1, \ldots, i\}$
- **Case 1:** Final interval is not in O ($i \notin O_i$) $\Rightarrow O_i = O_{i-1}$
- **Case 2:** Final interval is in O ($i \in O$) $\Rightarrow O_i = \{i\} + O_{p(i)}$

- $OPT(i) = \max\{\underbrace{OPT(i-1)}_{\text{then } O_i = O_{i-1}}, \underbrace{v_i + OPT(p(i))}_{\text{then } O_i = \{i\} + O_{p(i)}}\}$

Interval Scheduling: Take II

$O = \{5, 3, 1\}$



$M[i] = OPT(i)$

M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	~~M[6]~~
0	2	4	6	7	8	8

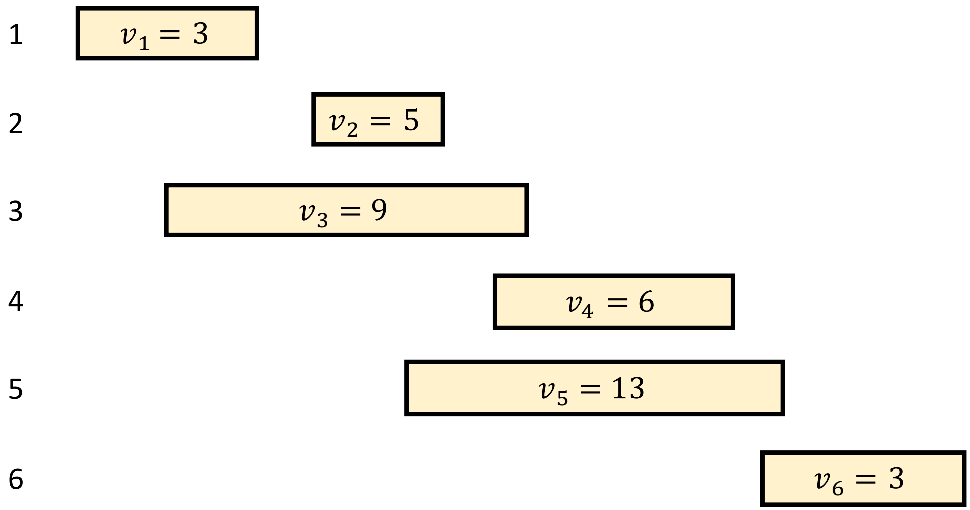
Interval Scheduling: Take III

```
// All inputs are global vars
FindSched(M,n):
  if (n = 0): return ∅
  elseif (n = 1): return {1}
  elseif (vn + M[p(n)] > M[n-1]):
    return {n} + FindSched(M,p(n))
  else:
    return FindSched(M,n-1)
```

- What is the running time of **`FindSched(n)`**?

$O(n)$ time

Now You Try



$p(1) = 0$

$p(2) = 1$

$p(3) = 0$

$p(4) = 2$

$p(5) = 1$

$p(6) = 4$

M[0]	M[1]	M[2]	M[3]	M[4]	M[5]	M[6]

Dynamic Programming Recap

- Express the optimal solution as a **recurrence**
  - Identify a small number of subproblems
  - Relate the optimal solution on subproblems
- Efficiently solve for the **value** of the optimum
  - Simple implementation is exponential time
  - Top-Down: store solution to subproblems
  - Bottom-Up: iterate through subproblems in order
- Find the **solution** using the table of **values**