CS3000: Algorithms & Data
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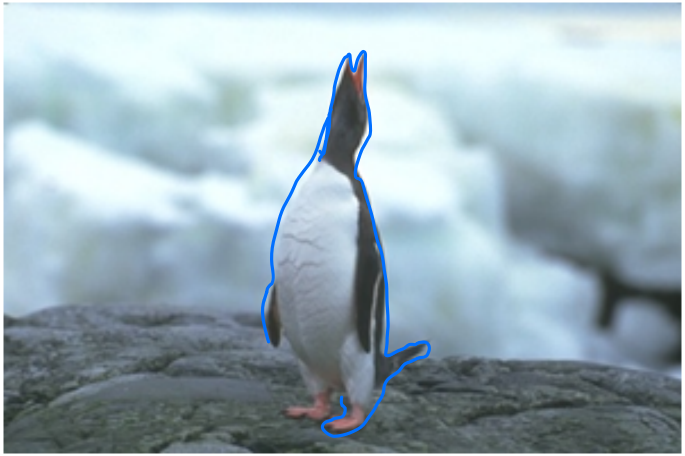
Lecture 21:
More Applications of Network Flow

Nov 30, 2018

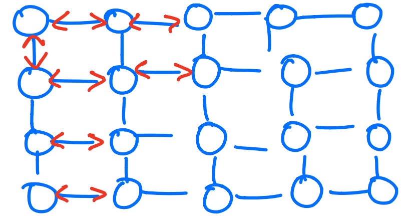
Image Segmentation

Image Segmentation



- Separate image into foreground and background
- We have some idea of:
  - whether pixel i is in the foreground or background
  - whether pair (i,j) are likely to go together

Image Segmentation



- Input:
  - a directed graph $G = (V, E)$; V = "pixels", E = "pairs"
  - likelihoods $a_i, b_i \geq 0$ for every $i \in V$
  - separation penalty $p_{ij} \geq 0$ for every $(i,j) \in E$

a = foreground
b = background

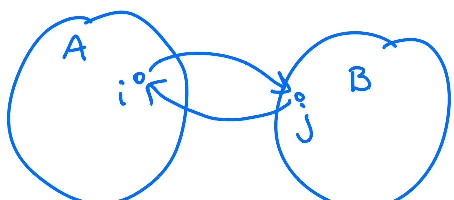
- Output:

"foreground" "background"

  - a partition of V into (A, B) that maximizes

$$q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

"quality"



Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG asks us to maximize, MINCUT asks us to minimize

$$\max_{A,B} \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij} \quad \Longleftrightarrow \quad \min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

  - SEG allows any partition, MINCUT requires $s \in A$, $t \in B$

Reduction to MinCut

- How should the reduction work?
  - capacity of the cut should correspond to the function we're trying to minimize

$$\boxed{\min_{\substack{A,B \\ s\in A \\ t\in B}} \sum_{i\in A} b_i + \sum_{j\in B} a_j + \sum_{\substack{(i,j)\in E \\ \text{from } A \text{ to } B}} p_{ij}}$$

$$\|$$

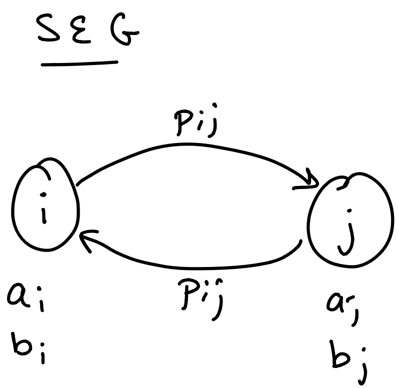
$$\boxed{\min_{\substack{A,B \\ s\in A \\ t\in B}} \sum_{\substack{(i,j)\in E \\ \text{from } A \text{ to } B}} c(i,j)}$$

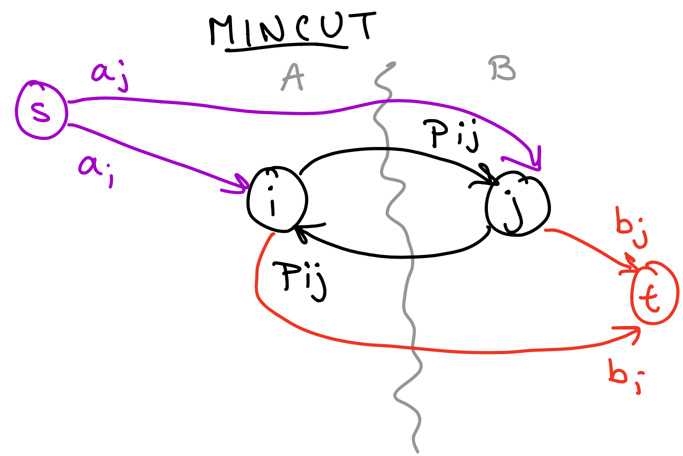
choose a flow network and capacities s.t.

Reduction to MinCut

- How should the reduction work?
  - capacity of the cut should correspond to the function we're trying to minimize

$$\min_{A,B} \sum_{i\in A} b_i + \sum_{j\in B} a_j + \sum_{\substack{(i,j)\in E \\ \text{from } A \text{ to } B}} p_{ij}$$





Reduction to MinCut

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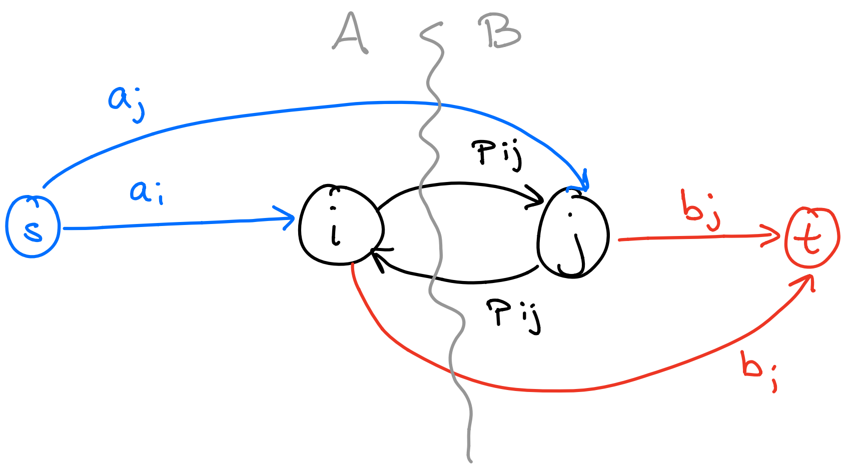
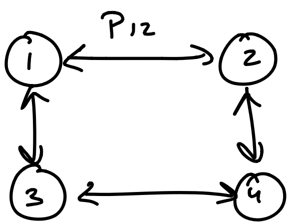
  - capacity of the cut should correspond to the function we're trying to minimize

$$\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

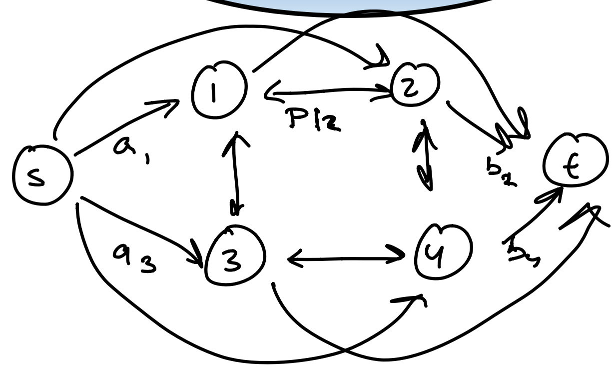


Step 1: Transform the Input

Input G,{a,b,p} for SEG

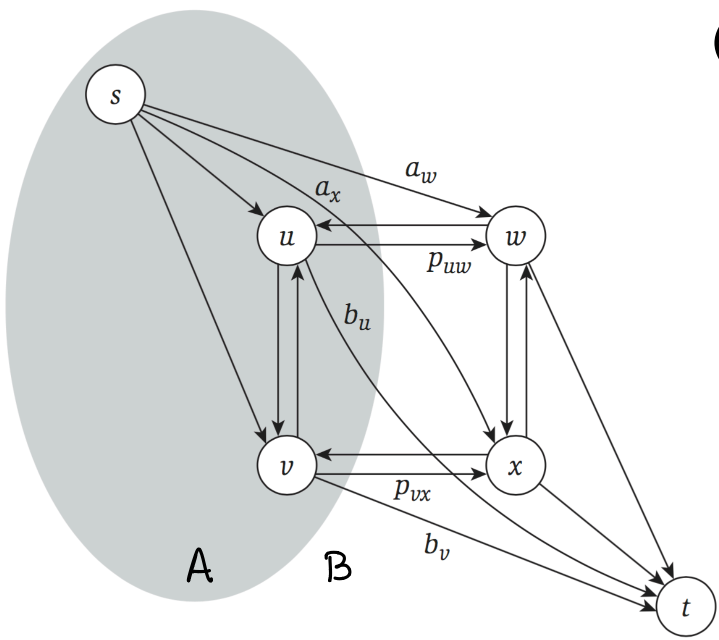


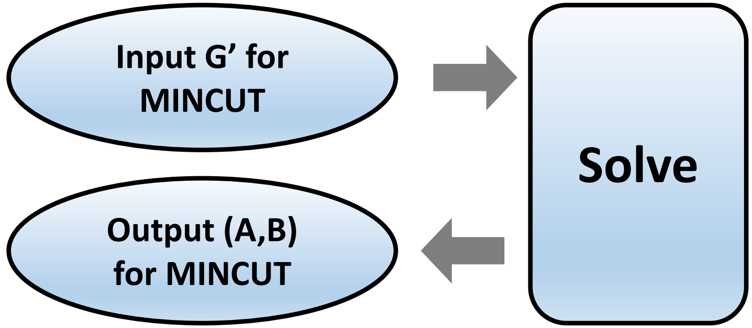
Input G' for MINCUT



time: $O(m+n)$

Step 2: Receive the Output



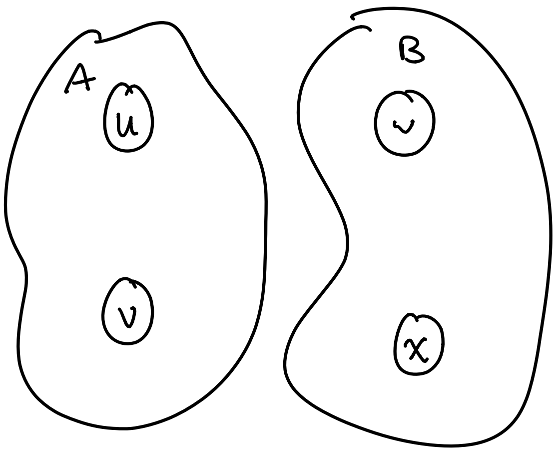


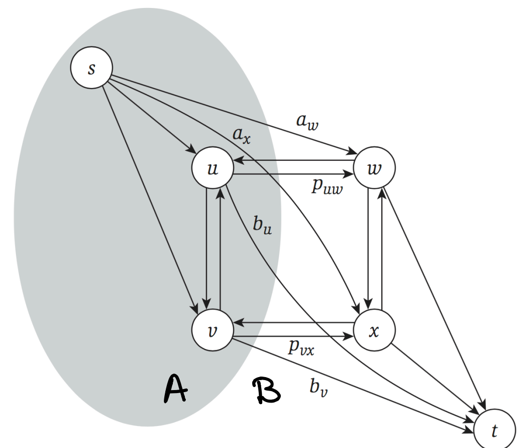
time: time to solve min cut in a graph with $n+2$ nodes and $2n+m$ edges

$O(mn)$

Step 3: Transform the Output

Output (A,B) for SEG ← **Output (A,B) for MINCUT**





time: $O(n)$

Reduction to MinCut

- correctness?

$$\max_{A,B} q(A,B)$$

All the work is here

$$= \min_{A,B} -q(A,B)$$

$$= \min_{A,B} cap_G(A \cup \{s\}, B \cup \{t\})$$

- running time?

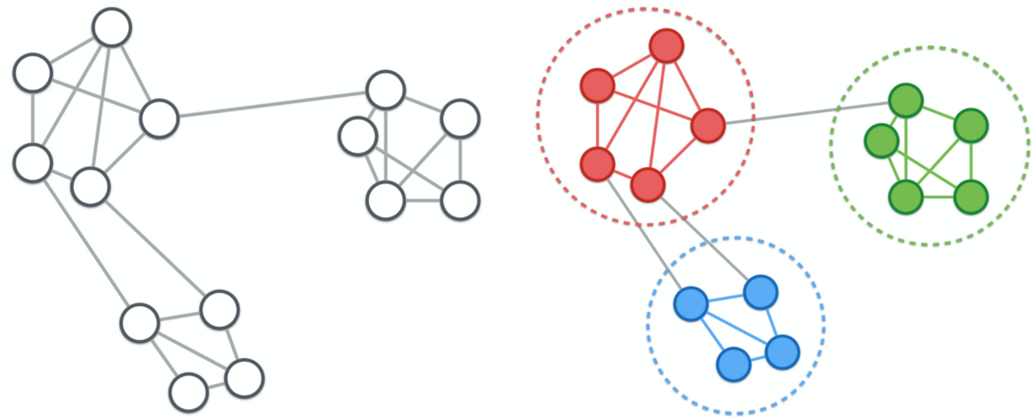
$$O(m+n) + O(mn) + O(n)$$

transform the input   solve mincut   transform the output

$$= O(mn)$$

Densest Subgraph

Image Segmentation



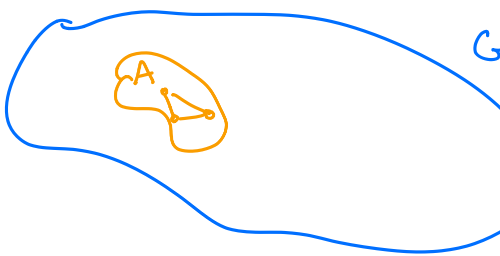
- Want to identify communities in a network
  - “Community”: a set of nodes that have a lot of connections inside and few outside

Densest Subgraph

- Input:
  - an undirected graph $G = (V, E)$
- Output:
  - a subset of nodes $A \subseteq V$ that maximizes $\frac{2|E(A,A)|}{|A|}$

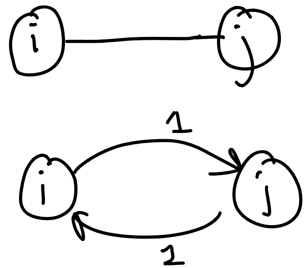
$$E(A,B) = \{ (i,j) \in E \;\; i \in A,\ j \in B \}$$

if | maximize $\frac{|E(A,A)|}{\binom{|A|}{2}}$



Reduction to MinCut

- Different objectives
  - find (A, B) to maximize $\frac{2|E(A,A)|}{|A|}$
  - find (A, B) to minimize $|E(A, B)|$



MINCUT

- Suppose $\frac{2|E(A,A)|}{|A|} \geq \delta$ and see what that implies

$$\Leftrightarrow 2|E(A, A)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in A} \deg(v) \; - |E(A, B)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in V} \deg(v) \; - \Sigma_{v\in B} \deg(v) - |E(A, B)| \geq \delta|A|$$

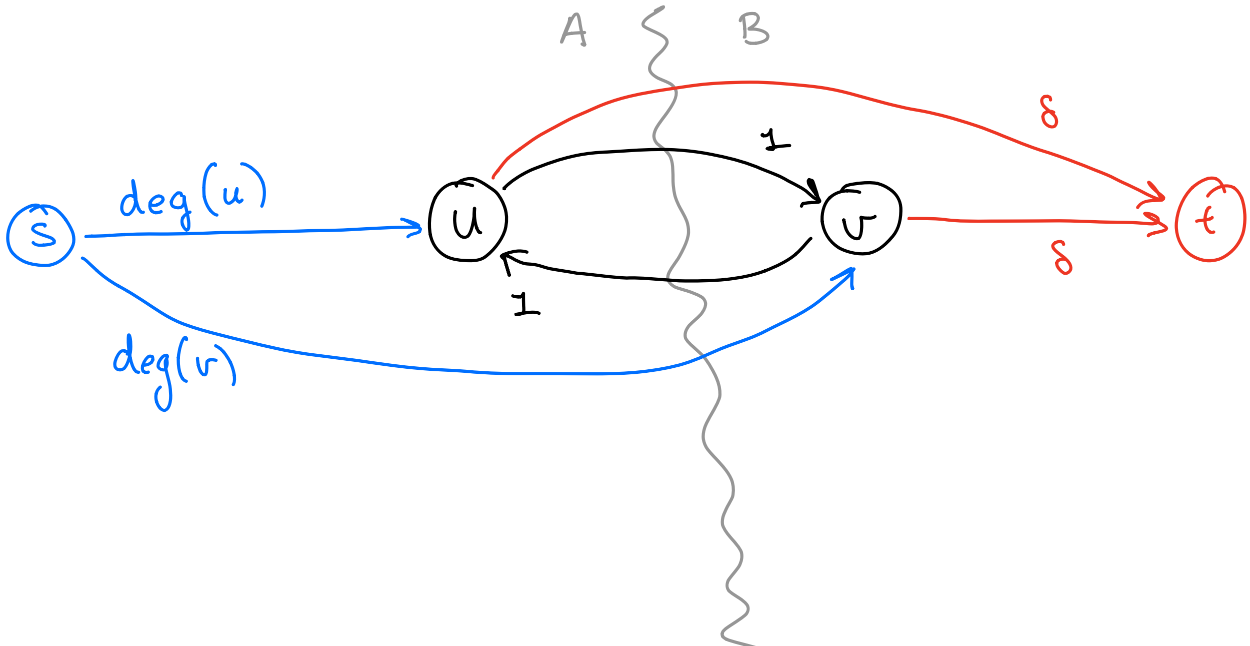
$$\Leftrightarrow 2|E| \; - \Sigma_{v\in B} \deg(v) - |E(A, B)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in B} \deg(v) + \delta|A| + \; |E(A, B)| \leq 2|E|$$

$$\Leftrightarrow \sum_{v\in B} \deg(v) + \sum_{v\in A} \delta + \sum_{(u,v)\in E \text{ from } A \text{ to } B} 1 \leq 2|E|$$

Reduction to MinCut

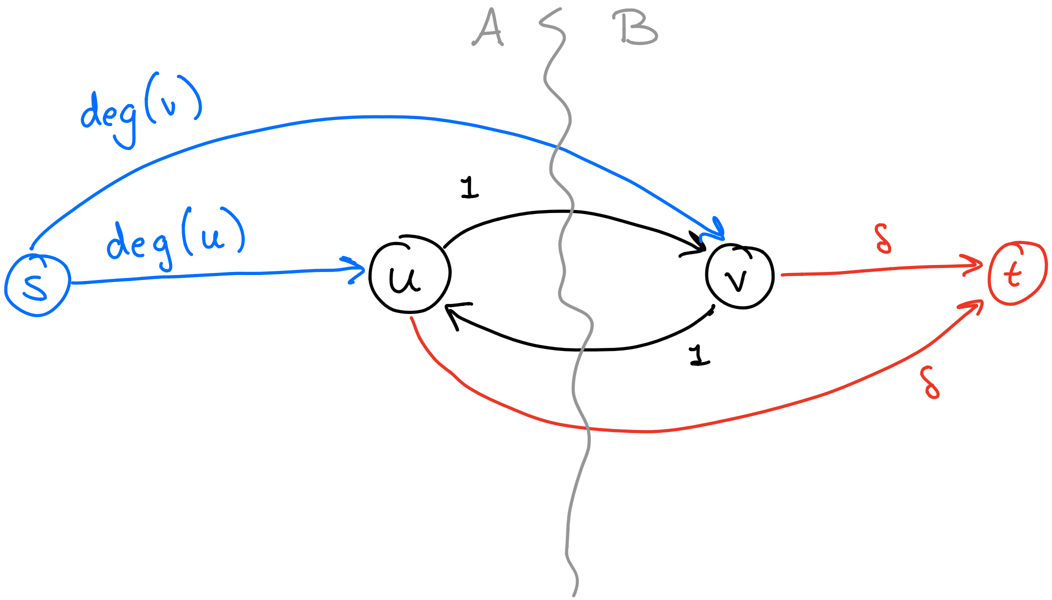
$$\boxed{\sum_{v \in B} \deg(v) + \delta|A| + |E(A,B)| \le 2|E|}$$



$\left(\text{mincut in } G\right) \le 2|E|$ if and only if $\exists A \quad \frac{2|E(A,A)|}{|A|} \geq \delta$

Reduction to MinCut

$$\boxed{\sum_{v \in B} \deg(v) + \delta|A| + |E(A,B)| \le 2|E|}$$

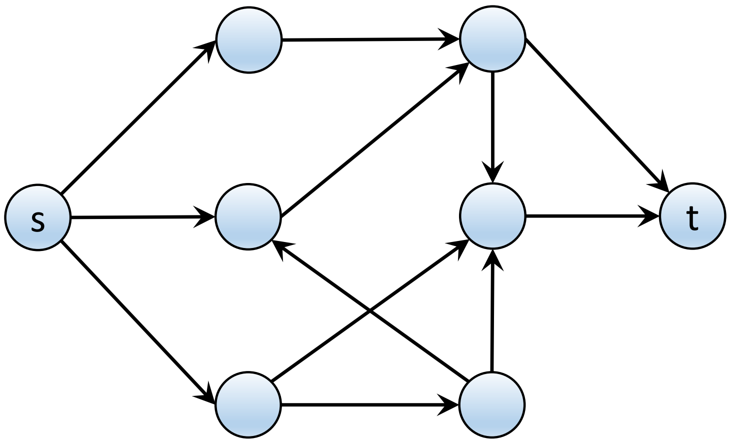


Edge-Disjoint Paths

(Edge) Disjoint Paths

- **Input:** directed graph $G = (V, E, s, t)$
- **Output:** a largest set of edge-disjoint s-t paths
  - A set of s-t paths $P_1, \ldots, P_k$ is edge disjoint if the paths do not share any edges

A large set of disjoint paths means we can tolerate edge failures.



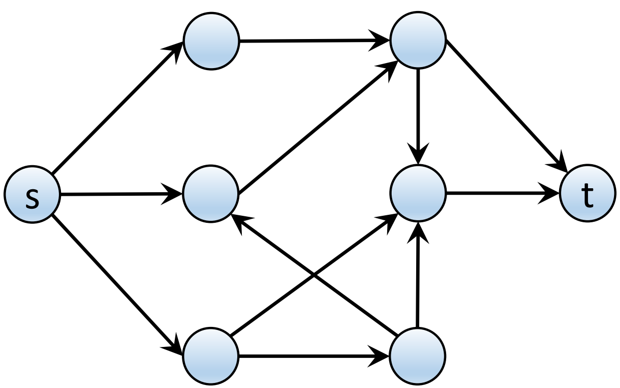
Bipartite Matching

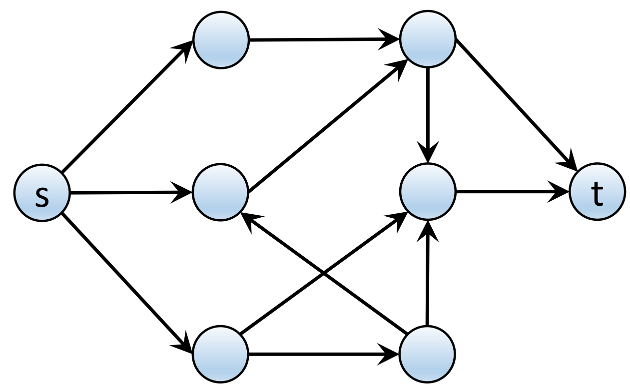
- There is a reduction that uses integer maximum s-t flow to solve edge disjoint paths.

Step 1: Transform the Input

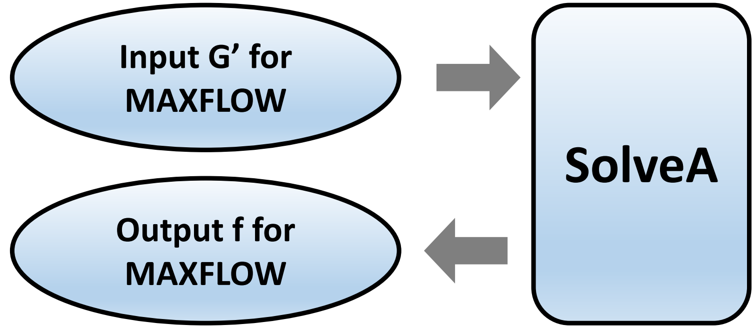
Input G for EDP

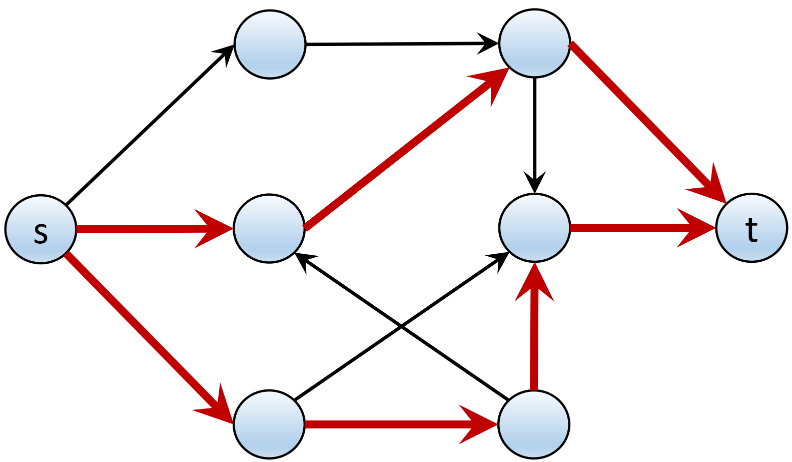
Input G’ for MAXFLOW





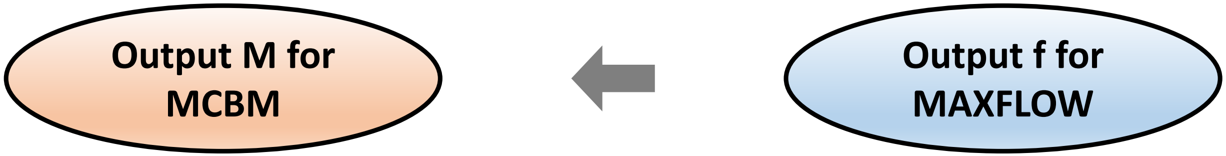
Step 2: Receive the Output





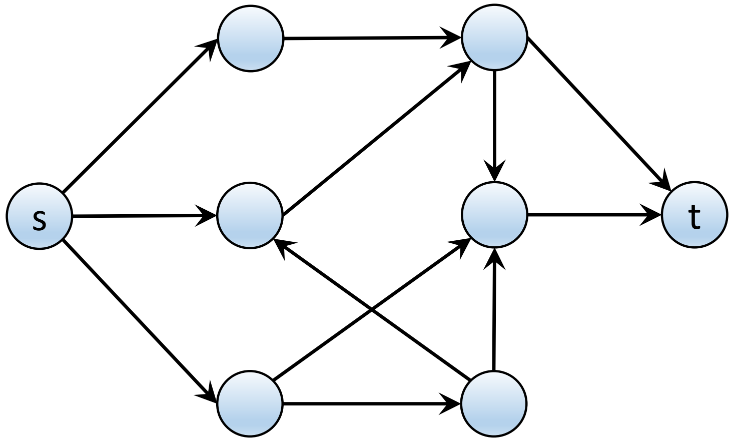
Red arrow means f(e)=1
Black means f(e) = 0

Step 3: Transform the Output



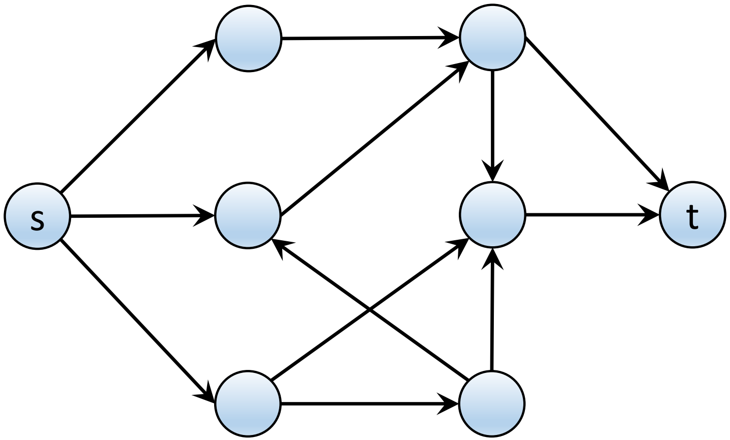
Correctness

- **Easy Direction:** If there are k edge disjoint paths then there is a flow of value k



Correctness

- **Harder Direction:** If there is a flow of value k, then there are k edge disjoint paths



Running Time

- Need to analyze the time for:
  - (1) Producing G’ given G
  - (2) Finding a maximum flow in G’
  - (3) Producing M given G’

Summary

Solving maximum s-t flow in a graph with n nodes and m edges and c(e)=1 in time T

Solving edge disjoint paths in a graph with n nodes and m edges in time T + O(mn)

- Can solve edge disjoint paths in time O(nm) using Ford-Fulkerson

(Node) Disjoint Paths

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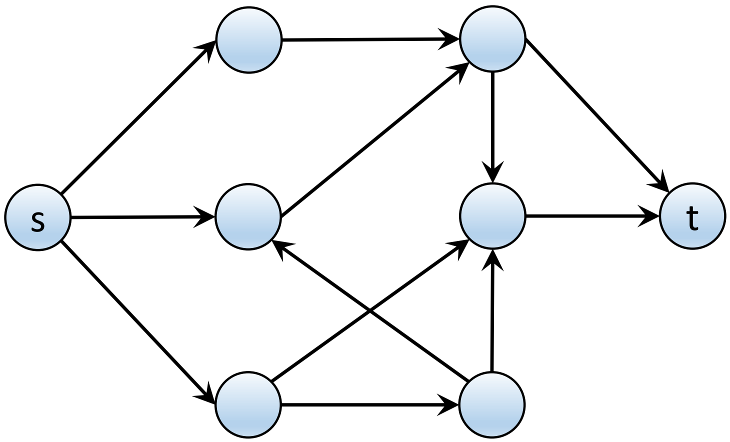

Step 1: Transform the Input