CS3000: Algorithms & Data
Jonathan Ullman

Lecture 19:

- Midterm II Review

Nov 13, 2018

Topics to Review

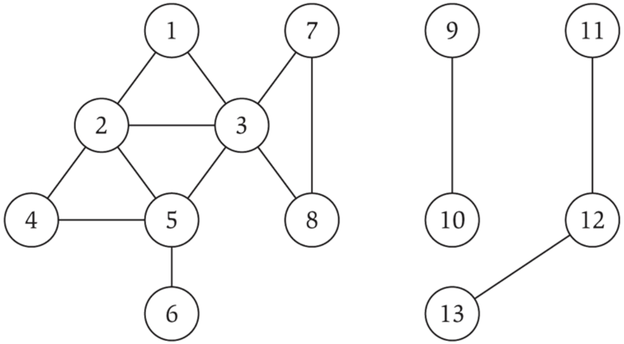
- Key Graph Definitions / Properties
  - Directed/Undirected
  - Weighted/Unweighted
  - Trees, DAGs
  - Paths, Cycles
  - Connected Components, Strongly Connected Components

Graphs: Key Definitions

$|V| = n \quad |E| = m$

- **Definition:** A directed graph $G = (V, E)$
  - V is the set of nodes/vertices
  - $E \subseteq V \times V$ is the set of edges
  - An edge is an ordered $e = (u, v)$ "from u to v"
- **Definition:** An undirected graph $G = (V, E)$
  - Edges are unordered $e = (u, v)$ "between u and v"
- **Simple Graph:**
  - No duplicate edges
  - No self-loops $e = (u, u)$

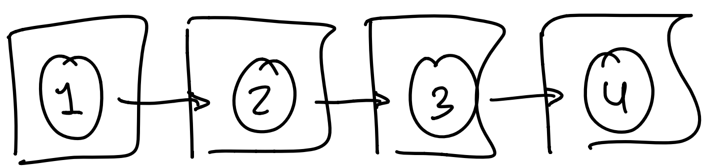
$m = O(n^2)$



Paths/Connectivity

- A path is a sequence of consecutive edges in E
  - $P = \{(u, w_1), (w_1, w_2), (w_2, w_3), \dots, (w_{k-1}, v)\}$
  - $P = u - w_1 - w_2 - w_3 - \cdots - w_{k-1} - v$
  - The length of the path is the # of edges

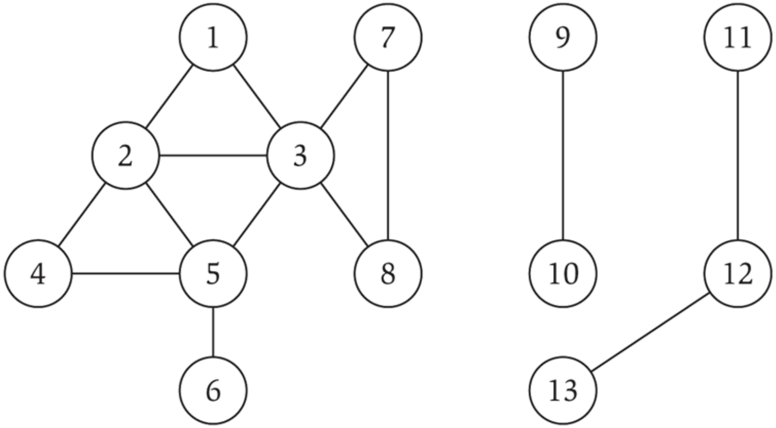
- An undirected graph is connected if for every two vertices $u, v \in V$, there is a path from u to v
- A directed graph is strongly connected if for every two vertices $u, v \in V$, there are paths from u to v and from v to u



connected, not strongly connected

Cycles

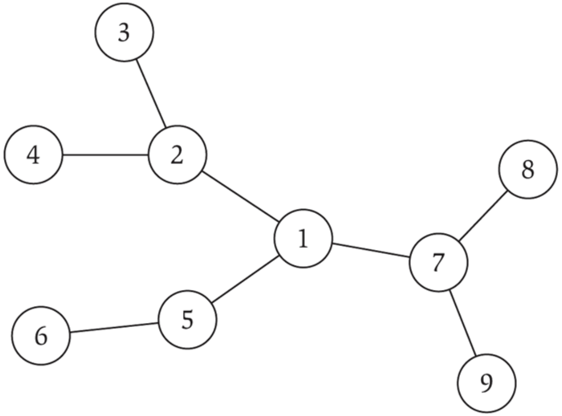
- A cycle is a path $v_1 - v_2 - \cdots - v_k - v_1$ where $k \geq 3$ and $v_1, \ldots, v_k$ are distinct



Trees

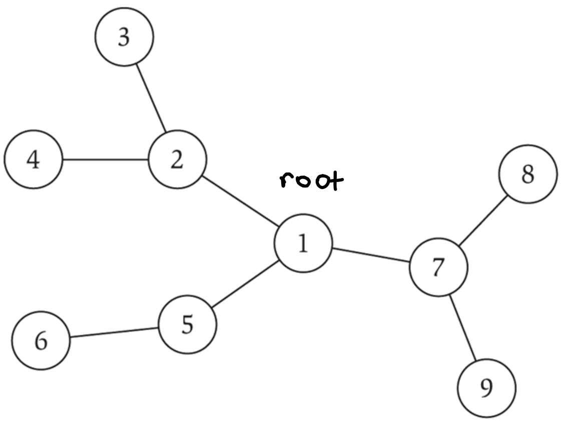
- A simple undirected graph G is a tree if:
  - G is connected
  - G contains no cycles

- **Theorem:** any two of the following implies the third
  - G is connected
  - G contains no cycles
  - G has $= n - 1$ edges

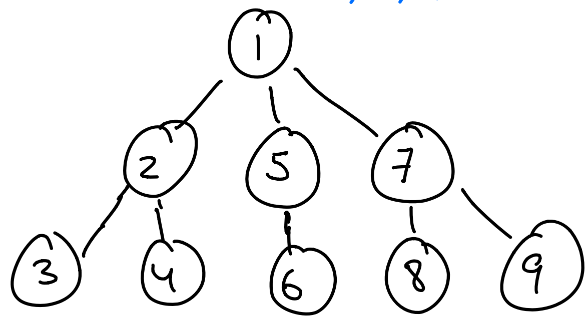


Trees

- **Rooted tree:** choose a root node r and orient edges away from r
  - Models **hierarchical structure**



"1 is the parent of 2, 5, 7"



"2, 5, 7 are the children of 1"

Topics to Review

- Graph Representations
  - Adjacency Matrix
  - Adjacency List

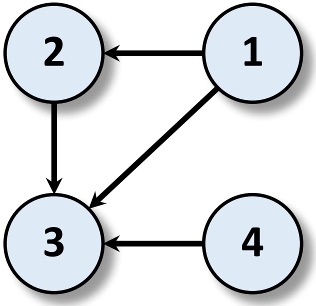
All algorithms we study are for the adjacency

Adjacency-Matrix Representation

- The adjacency matrix of a graph $G = (V, E)$ with n nodes is the matrix $A[1{:}n, 1{:}n]$ where

$$A[i,j] = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0



Cost

Space: $\Theta(n^2)$

Lookup (u,v): $\Theta(1)$ time

List Neighbors of u: $\Theta(n)$ time

Adjacency Lists (Directed)

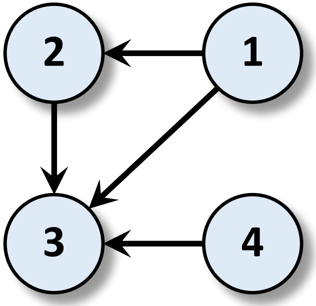
- The adjacency list of a vertex $v \in V$ are the lists
  - $A_{out}[v]$ of all u s.t. $(v, u) \in E$
  - $A_{in}[v]$ of all u s.t. $(u, v) \in E$

$A_{out}[1] = \{2,3\}$ $\quad$ $A_{in}[1] = \{\,\}$

$A_{out}[2] = \{3\}$ $\quad$ $A_{in}[2] = \{1\}$

$A_{out}[3] = \{\,\}$ $\quad$ $A_{in}[3] = \{1,2,4\}$

$A_{out}[4] = \{3\}$ $\quad$ $A_{in}[4] = \{\,\}$



Adjacency-List Representation

- The adjacency list of a vertex $v \in V$ is the list $A[v]$ of all the neighbors of v

$A[1] = \{2,3\}$

$A[2] = \{1,3\}$

$A[3] = \{1,2,4\}$

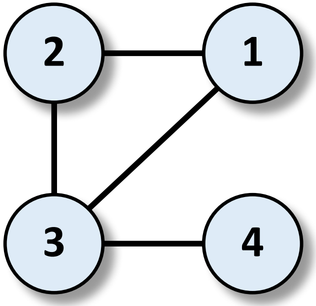
$A[4] = \{3\}$

Cost

Space: $\Theta(n + m)$

Lookup (u,v): $\Theta(\deg(u) + 1)$ time

List Neighbors of u: $\Theta(\deg(u) + 1)$ time



Topics to Review

- Finding (short) paths in graphs
  - BFS for finding:
    - Connected components
    - Strongly connected components
    - Shortest paths in unweighted graphs (i.e. fewest hops)
  - Dijkstra's algorithm for finding:
    - Shortest paths in graphs with non-negative lengths
  - Bellman-Ford algorithm for finding:
    - Shortest paths in graphs with negative lengths (no neg cycles)
    - Negative cycles if they exist
  - Structural properties of shortest paths
    - Dynamic programming
    - Shortest path trees

$\forall (u,v) \in E$

$d(s,v) \leq d(s,u) + \ell(u,v)$

BFS

- Informal Description: start at s, find all neighbors of s, find all neighbors of neighbors of s, …

- BFS Algorithm:
  - $L_0 = \{s\}$
  - $L_1 =$ all neighbors of L_0
  - $L_2 =$ all neighbors of L_1 that are not in L_0, L_1
  - …
  - $L_d =$ all neighbors of L_{d-1} that are not in $L_0, \dots, L_{d-1}$
  - Stop when L_{d+1} is empty.

Breadth-First Search Implementation

```
BFS(G = (V,E), s):
  Let found[v] ← false ∀v, found[s] ← true
  Let layer[v] ← ∞ ∀v, layer[s] ← 0
  Let i ← 0, L_0 = {s}, T ← ∅

  While (L_i is not empty):
    Initialize new layer L_{i+1}
    For (u in L_i):
      For ((u,v) in E):
        If (found[v] = false):
          found[v] ← true, layer[v] ← i+1
          Add (u,v) to T and add v to L_{i+1}
    i ← i+1
```

Implementing Dijkstra

```
Dijkstra(G = (V,E,{ℓ(e)}, s):
  d[s] ← 0, d[u] ← ∞ for every u != s
  parent[u] ←⊥ for every u
  Q ← V                    // Q holds the unexplored nodes

  While (Q is not empty):
    u ← argmin_{w∈Q} d[w]  //Find closest unexplored
    Remove u from Q

    // Update the neighbors of u
    For ((u,v) in E):
      If (d[v] > d[u] + ℓ(u,v)):
        d[v] ← d[u] + ℓ(u,v)
        parent[v] ← u

  Return (d, parent)
```

Recurrence

- **Subproblems:** $\mathrm{OPT}(v, j)$ is the length of the shortest $s \rightsquigarrow v$ path with at most j hops
- **Case u:** (u, v) is final edge on the shortest $s \rightsquigarrow v$ path with at most j hops

Recurrence:

$$\mathrm{OPT}(v, j) = \min\left\{\mathrm{OPT}(v, i-1), \min_{(u,v)\in E}\{\mathrm{OPT}(u, i-1) + \ell_{u,v}\}\right\}$$

$\mathrm{OPT}(s, j) = 0$ for every j

$\mathrm{OPT}(v, 0) = \infty$ for every v

Implementation (Bottom Up)

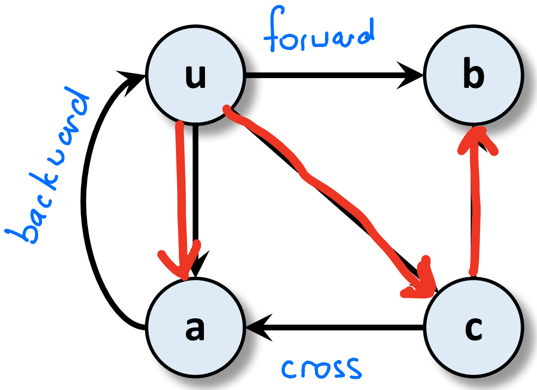
```
Shortest-Path(G, s)
   foreach node v ∈ V
      M[0,v] ← ∞
      P[0,v] ← ϕ
   M[0,s] ← 0

   for i = 1 to n-1
      foreach node v ∈ V
        M[i,v] ← M[i-1,v]
        P[i,v] ← P[i-1,v]
        foreach edge (v, w) ∈ E
           if (M[i-1,w] + ℓwv < M[i,v])
              M[i,v] ← M[i-1,w] + ℓwv
              P[i,v] ← w
```

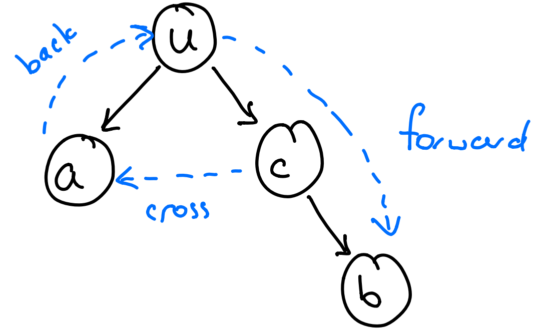
Topics to Review

- Depth-First Search
  - Types of edges (tree, forward, backward, cross)
  - Post-ordering
- Topological Sort
  - Fast algorithm using DFS
- Other graph algorithms
  - 2-coloring

Depth-First Search

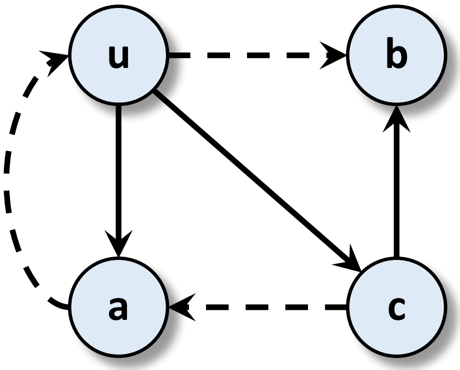
```
G = (V,E) is a graph
explored[u] = 0 ∀u

DFS(u):
  explored[u] = 1

  for ((u,v) in E):
    if (explored[v]=0):
      parent[v] = u
      DFS(v)
```





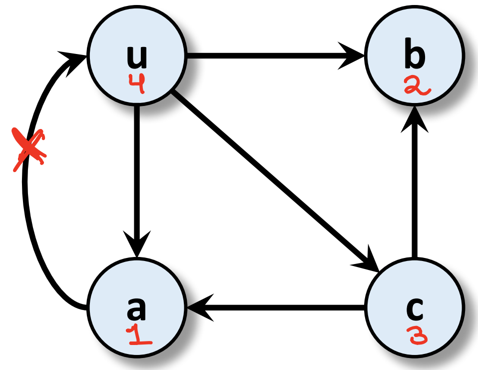
Depth-First Search

- **Fact:** The parent-child edges form a (directed) tree
- **Each edge has a type:**
  - **Tree edges:** $(u, a), (u, c), (c, b)$
    - These are the edges that explore new nodes
  - **Forward edges:** (u, b)
    - Ancestor to descendant (but not a child)
  - **Backward edges:** (a, u)
    - Descendant to ancestor
  - **Cross edges:** (c, a)
    - No ancestral relation



Post-Ordering

```
G = (V,E) is a graph
explored[u] = 0 ∀u

DFS(u):
  explored[u] = 1

  for ((u,v) in E):
    if (explored[v]=0):
      parent[v] = u
      DFS(v)

  post-visit(u)
```

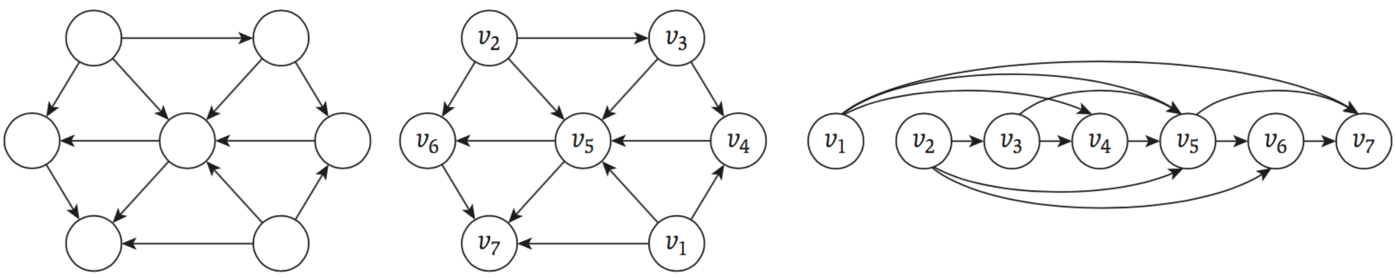


Vertex	Post-Order
u	4
a	1
b	2
c	3

- Maintain a counter **`clock`**, initially set **`clock = 1`**
- **`post-visit(u):`**
  `set postorder[u]=clock, clock=clock+1`

Directed Acyclic Graphs (DAGs)

- **DAG:** A directed graph with no directed cycles
- DAGs represent precedence relationships



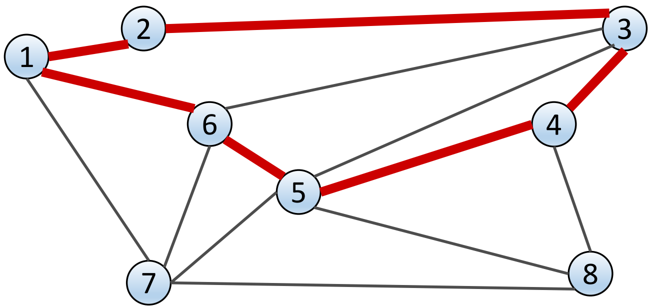
- A **topological ordering** of a directed graph is a labeling of the nodes from $v_1, \ldots, v_n$ so that all edges go "forwards", that is $(v_i, v_j) \in E \Rightarrow j > i$
  - **G has a topological ordering $\Leftrightarrow$ G is a DAG**
  - The reverse of a post-order is a topological order

Topics to Review

- Minimum Spanning Trees
  - Cut Property / Cycle Property
  - Four Algorithms:
    - Boruvka
    - Prim
    - Kruskal
    - Anti-Kruskal

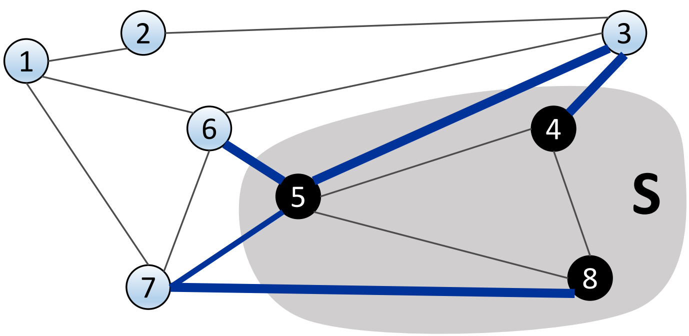
Cycles and Cuts

- Cycle: a set of edges $(v_1, v_2), (v_2, v_3), \dots, (v_k, v_1)$



Cycle C = (1,2),(2,3),(3,4),(4,5),(5,6),(6,1)

- Cut: a subset of nodes S



Cut S = {4, 5, 8}
Cutset = (5,6), (5,7), (3,4), (3,5), (7,8)

Properties of MSTs

- Assume all edge weights are distinct.
- Cut Property: Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e
  - We call such an e a safe edge
- Cycle Property: Let C be a cycle. Let e be the maximum weight edge in C. Then the MST T^* does not contain e.
  - We call such an e a useless edge

MST Algorithms

- There are at least four reasonable MST algorithms
  - Borůvka's Algorithm: start with $T = \emptyset$, in each round add cheapest edge out of each connected component
  - Prim's Algorithm: start with some s, at each step add cheapest edge that grows the connected component
  - Kruskal's Algorithm: start with $T = \emptyset$, consider edges in ascending order, adding edges unless they create a cycle
  - Reverse-Kruskal: start with $T = E$, consider edges in descending order, deleting edges unless it disconnects

Topics to Review

- Network Flow
  - Definitions (Flows, Cuts, Augmenting Path, Residual Graph)
  - Ford-Fulkerson Algorithm
    - Algorithm
    - Correctness
    - Running time analysis
    - Methods for choosing good augmenting paths (but not proofs)
  - MaxFlow-MinCut Theorem

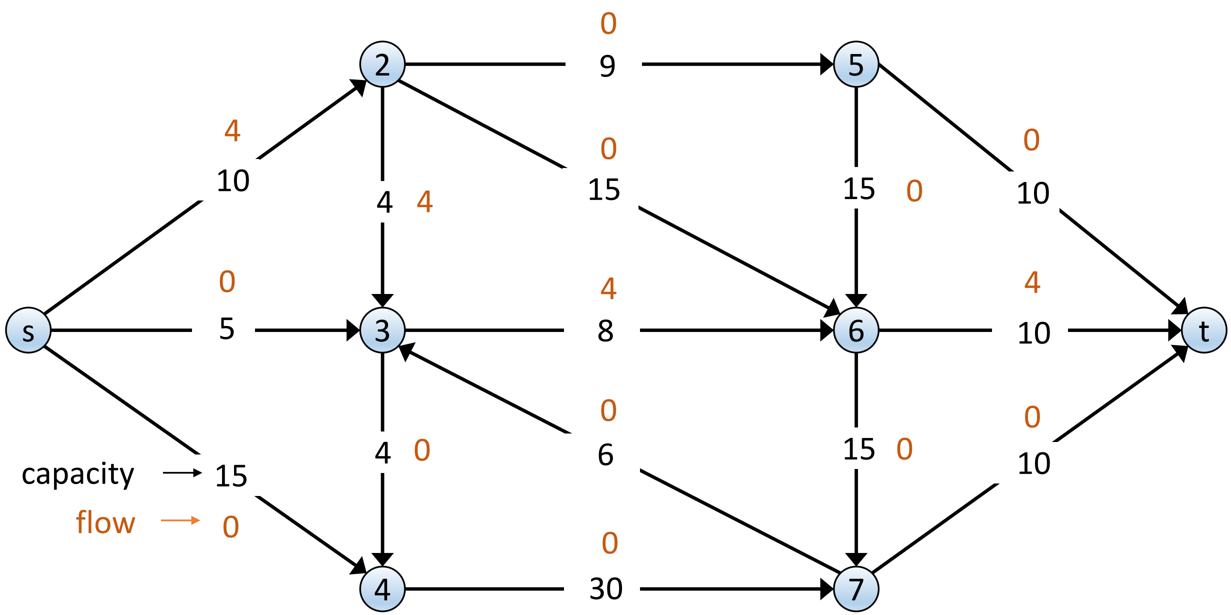
$$\max_{f} \text{val}(f) = \min_{A,B} \text{cap}(A,B)$$

- Can find a max flow in $O(mn)$ time.

Flows

- An s-t flow is a function $f(e)$ such that
  - For every $e \in E$, $0 \leq f(e) \leq c(e)$ (capacity)
  - For every $v \in E$, $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

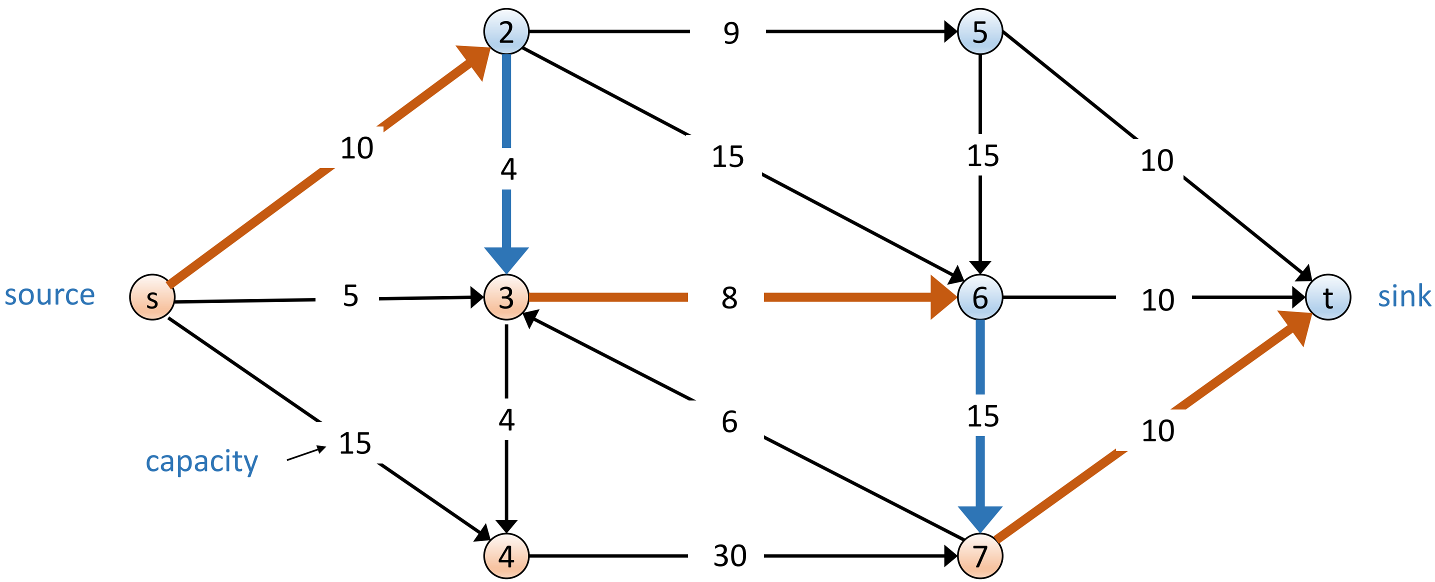
- The value of a flow is $val(f) = \sum_{e \text{ out of } s} f(e)$



Cuts

- An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$

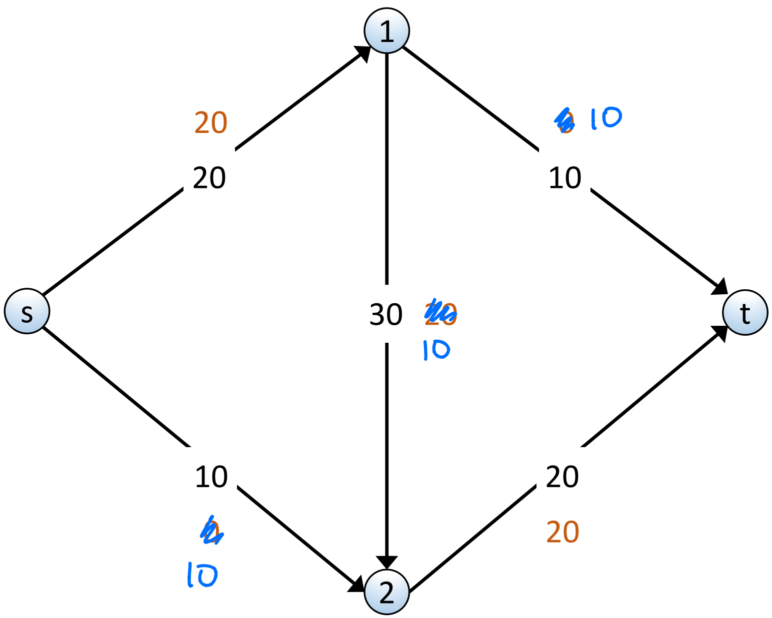
- The capacity of a cut (A,B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



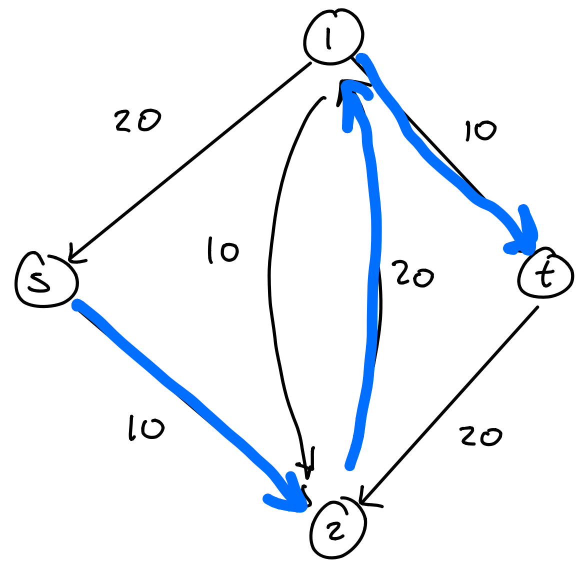
Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck

flow f



residual graph Gf



Ford-Fulkerson Algorithm

```
FordFulkerson(G,s,t,{c})
    for e ∈ E: f(e) ← 0
    Gf is the residual graph

    while (there is an s-t path P in Gf)
        f ← Augment(Gf,P)
        update Gf

    return f
```

```
Augment(Gf, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if e ∈ E:   f(e) ← f(e) + b
        else:       f(e) ← f(e) - b
    return f
```

Review Problems

Review Problem

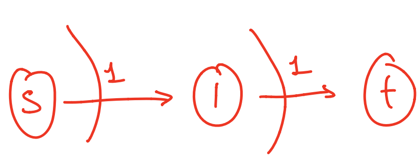
Given a flow network $G = (V, E, s, t, \overbrace{\{c(e)\}}^{\text{integer}})$, and a maximum flow f^*, output a list of all edges e, s.t. increasing $c(e)$ by 1 increases the max flow.
Run in time $O(m+n)$.

Candidate Algorithm:

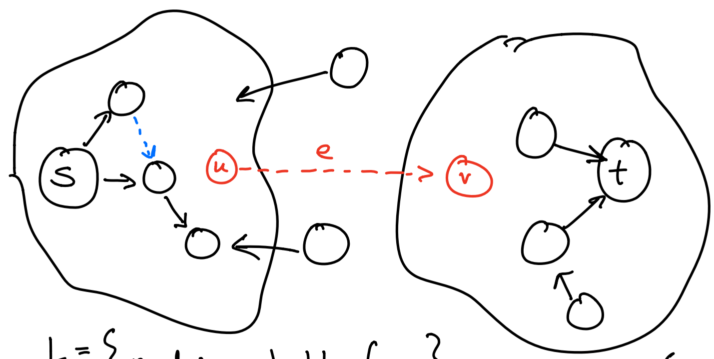
- Find the min cut A^*, B^*
- Output all edges crossing from A^* to B^*

"Proof":

- By duality $\max_f \text{val}(f) = \min_{A,B} \text{cap}(A,B)$
- Increasing $c(e)$ for any edge from A^* to B^*, increases capacity of the min cut.
- Therefore it increases the value of the max flow.



G_{f^*}



$e = (u,v)$

$L = \{\text{nodes reachable from } s\}$

$R = \{\text{nodes that can reach } t\}$

- If $f^*(e) < c(e)$, then adding to $c(e)$ can't create an augmenting path

- If $f^*(e) = c(e)$, then adding to $c(e)$ will add the edge e back to the residual graph

  - adding $e = (u,v)$ to the residual graph creates an augmenting path iff

    ① u is reachable from s in G_{f^*}

    ② t is reachable from v in G_{f^*}

Pseudocode

① Let G_{f^*} be the residual graph, and $G^R_{f^*}$ be the reverse of G_{f^*}

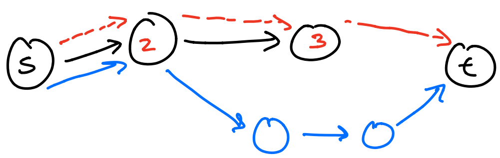
② Let L, R be [above]

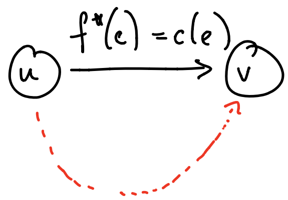
③ let $S = \emptyset$

④ For ($(u,v) \in E$):
    If ($u \in L \land v \in R$):
        add (u,v) to S

⑤ Return S

- If I decrease $c(e)$ by 1 and $f^*(e) = c(e)$ then f^* is no longer a legal flow.
- We can "fix" the flow by choosing any path through the edge e s.t. all edges on the path carry flow





$P \rightarrow \neg Q$

$P \leftarrow \neg Q$

- suppose there is a $u \rightarrow v$ path in G_{f^*} then decreasing capacity of e by 1 will not reduce the max flow

Bonus Review Problem

- Prove the following by induction: in any rooted binary tree, the number of nodes with exactly two children is one less than the number of leaves.

Review Problem #4

- Design an algorithm that takes an undirected $G = (V, E)$, and a pair of nodes s, t and outputs the number of shortest s-t paths in G.

Review Problem #5

- Design an algorithm to find a fattest s-t path in a flow network $G = (V, E, s, t, \{c(e)\})$

Review Problem #6

- There are n bank accounts $A_1, \dots, A_n$, and you are given m constraints of the form
  - A_i was closed before A_j opened
  - A_i and A_j were open (at least partially) simultaneously
- Design an algorithm to determine if there are opening and closing times for the accounts that satisfy all constraints

Review Problem #7

- Prove the following by contradiction: if G is an undirected simple graph with $2n$ nodes, and every node has degree $\geq n$, then G is connected.