CS3000: Algorithms & Data
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Lecture 16:
- Minimum Spanning Trees

Nov 2, 2018

Minimum Spanning Trees

Network Design

- **Build a cheap, well connected network** (= graph)
- We are given
  - a set of nodes $V = \{v_1, \dots, v_n\}$
  - a set of possible edges $E \subseteq V \times V$
- Want to build a network to connect these locations
  - Every v_i, v_j must be connected
  - Must be as cheap as possible
- Many variants of network design
  - Recall the bus routes problem from HW2

Minimum Spanning Trees (MST)

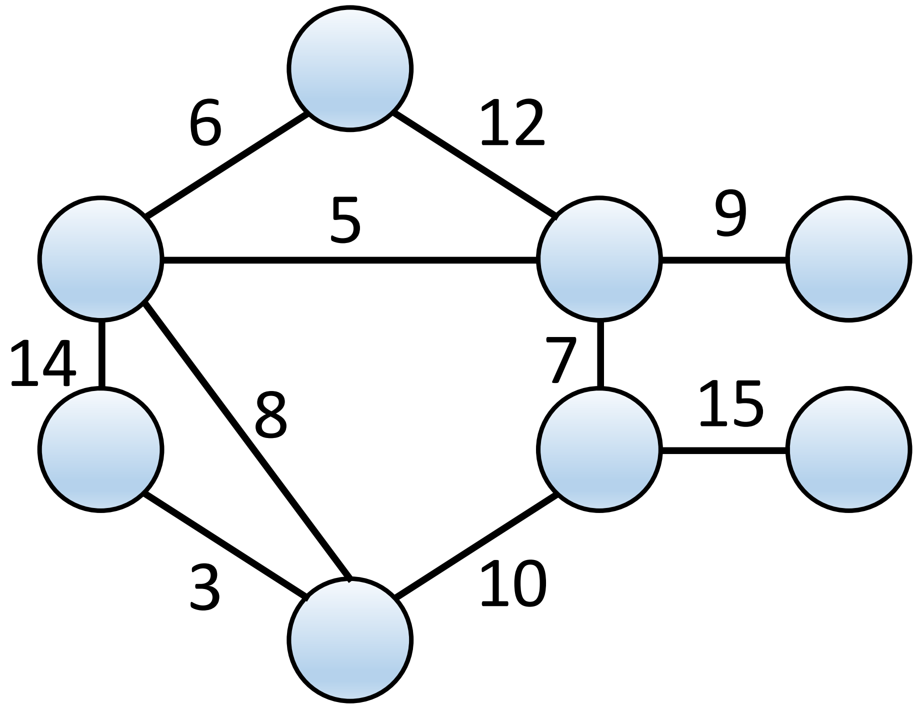
- **Input:** a weighted graph $G = (V, E, \{w_e\})$
  - Undirected, connected, weights may be negative
  - All edge weights are distinct (makes life simpler)

- **Output:** a spanning tree T of minimum cost
  - A spanning tree of G is a subset of $T \subseteq E$ of the edges such that (V, T) forms a tree (connected, acyclic)
  - Cost of a spanning tree T is the sum of the edge weights

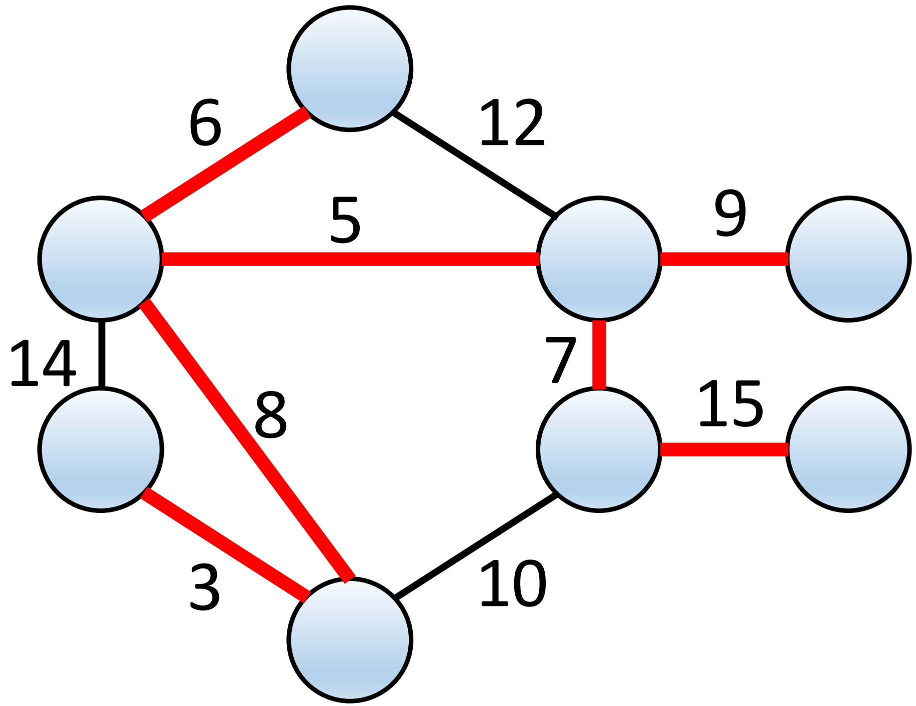
$$\text{cost}(T) = \sum_{e \in T} w(e)$$

MST: $T^* \in \operatorname*{argmin}_{\text{trees } T} \text{cost}(T)$

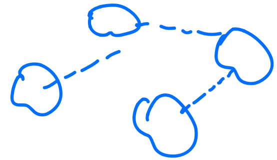
Minimum Spanning Trees (MST)



Minimum Spanning Trees (MST)



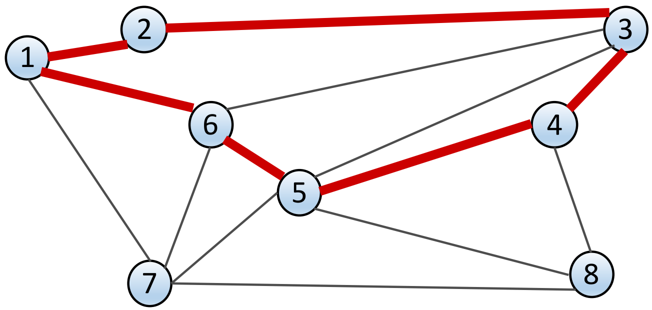
MST Algorithms



- There are at least four reasonable MST algorithms
  - Borůvka’s Algorithm: start with $T = \emptyset$, in each round add cheapest edge out of each connected component
  - Prim’s Algorithm: start with some s, at each step add cheapest edge that grows the connected component
  - Kruskal’s Algorithm: start with $T = \emptyset$, consider edges in ascending order, adding edges unless they create a cycle
  - Reverse-Kruskal: start with $T = E$, consider edges in descending order, deleting edges unless it disconnects

Cycles and Cuts

- **Cycle:** a set of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1)$

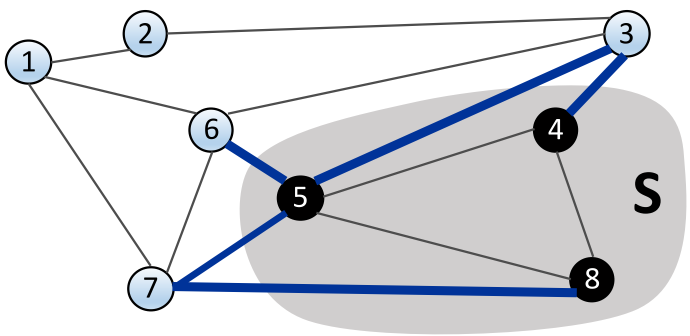


Cycle C = (1,2),(2,3),(3,4),(4,5),(5,6),(6,1)

- **Cut:** a subset of nodes S

$$\mathrm{Cutset}(S) = \left\{ (u,v) \in S : \begin{matrix} u \in S \\ v \notin S \end{matrix} \right\}$$

"Edges cut by S"



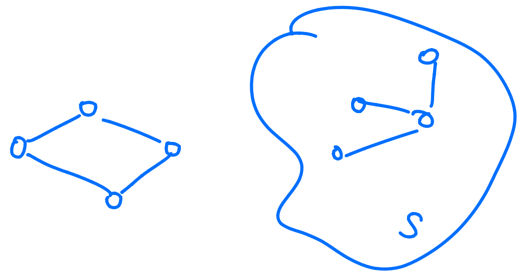
Cut S	=	{4, 5, 8}
Cutset	=	(5,6), (5,7), (3,4), (3,5), (7,8)

Cycles and Cuts

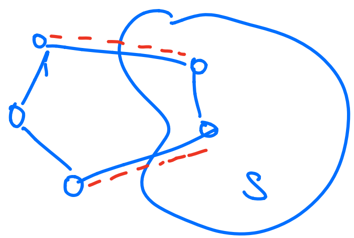
"Every time I leave S, I must come back."

- **Fact:** a cycle and a cutset intersect in an even number of edges

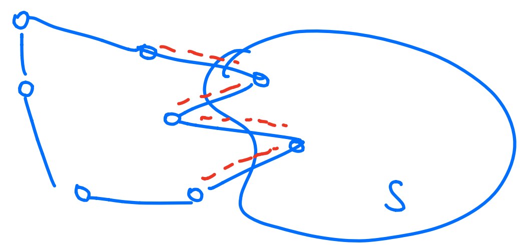
$|Cut \cap C| = 0$



$|Cut \cap C| = 2$



$|Cut \cap C| = 4$



Properties of MSTs

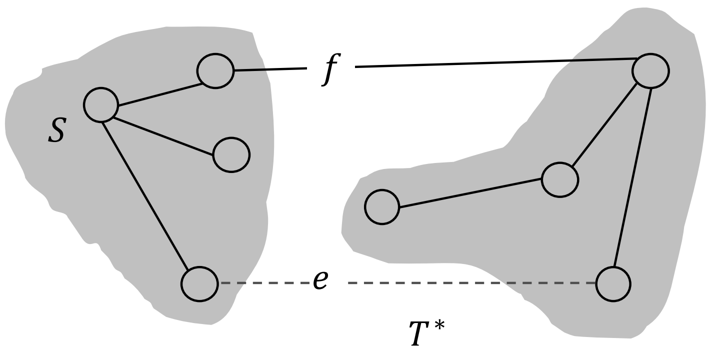
assumes that wts are distinct

- **Cut Property:** Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e
  - We call such an e a safe edge

- **Cycle Property:** Let C be a cycle. Let f be the maximum weight edge in C. Then the MST T^* does not contain f.
  - We call such an f a useless edge

Proof of Cut Property

- **Cut Property:** Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e

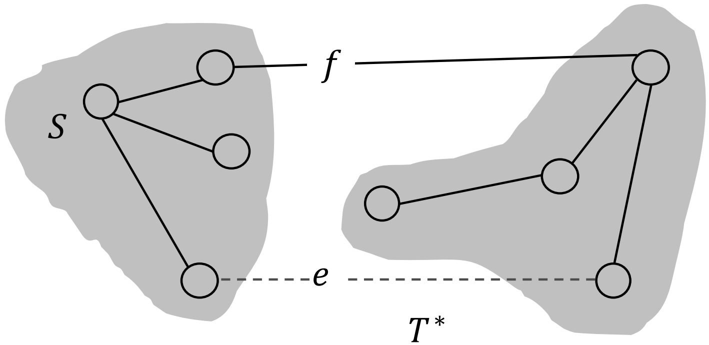


- Proof by Contradiction:
- Let T^* be an MST, $e \notin T^*$
- There is some $f \in T^*$ that is also in Cutset(S)
- $w(f) > w(e)$ because e is a safe edge for cut S

$\Rightarrow \text{cost}(T^* - \{f\} + \{e\}) < \text{cost}(T^*)$

Proof of Cut Property

- **Cut Property:** Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e

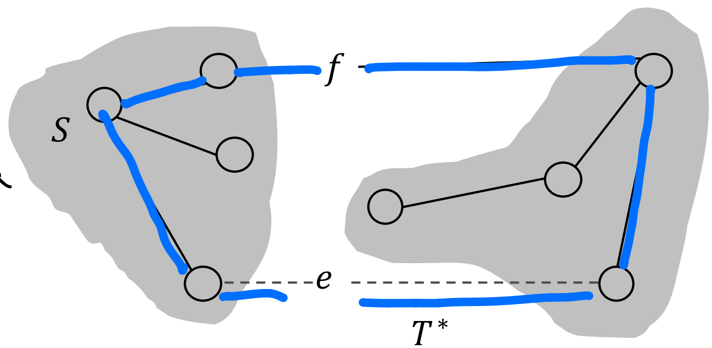


- $T^* - \{f\} + \{e\}$ is a spanning tree
  - $T^* - \{f\}$ has two connected components, S and S^c
  - e bridges S and S^c
- Then T^* is not an MST, contradiction. $\square$

Proof of Cycle Property

- **Cycle Property:** Let C be a cycle. Let f be the max weight edge in C. The MST T^* does not contain f.

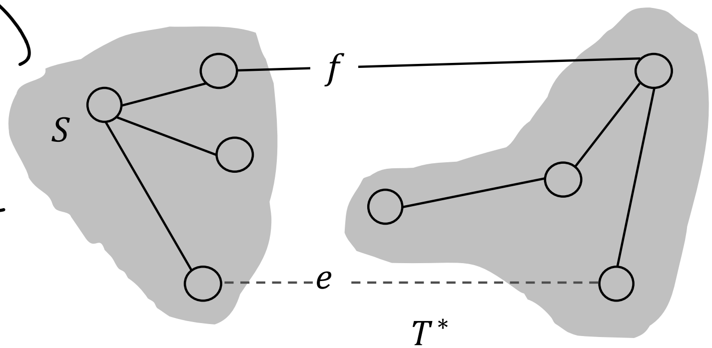
- Proof by contradiction:
- Assume T^* is an MST, $f \in T^*$
- $T^* - \{f\}$ has two connected components S, S^c
- C intersects Cutset(S) in an even # of edges.
  $\Rightarrow$ there is an $e \in C$ and $\in$ Cutset
  $\Rightarrow$ $w(e) < w(f)$



Proof of Cycle Property

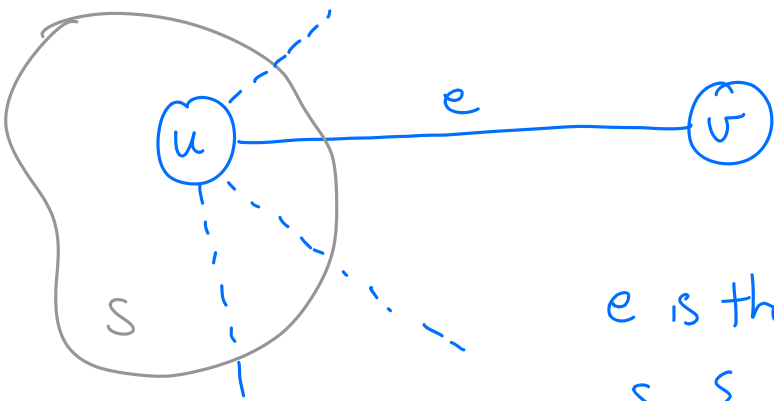
- **Cycle Property:** Let C be a cycle. Let f be the max weight edge in C. The MST T^* does not contain e.

- $\text{cost}(T^* - \{f\} + \{e\}) < \text{cost}(T^*)$
- $T^* - \{f\} + \{e\}$ is spanning tree
- But then T^* is not an MST, contradiction. $\square$



Ask the Audience

- Assume G has distinct edge weights
- **True/False?** If e is the edge with the smallest weight, then e is always in the MST T^*
- **True/False?** If e is the edge with the largest weight, then e is never in the MST T^*



e is the safe edge for $S = \{u\}$

Ask the Audience

- Assume G has distinct edge weights
- **True/False?** If e is the edge with the smallest weight, then e is always in the MST T^*
- **True/$\boxed{\textbf{False}}$?** If e is the edge with the largest weight, then e is never in the MST T^*

what if there is only one edge?

The "Only" MST Algorithm

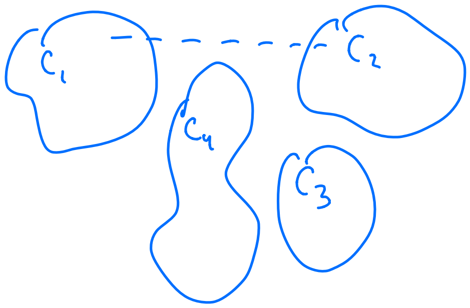
- **GenericMST:**
  - Let $T = \emptyset$
  - Repeat until T is connected:
    - Find one or more safe edges not in T
    - Add safe edges to T

- **Theorem: GenericMST** outputs an MST

Proof: ① We only add safe edges

② If T not connected, then there exists a safe edge

Suppose T is not connected



There must be edges btw each component or else E is not connected

$\Rightarrow$ there is some edge in the cut C_1

$\Rightarrow$ there is a safe edge in the cut C_1

Borůvka's Algorithm

- **Borůvka:**
  - Let $T = \emptyset$
  - Repeat until T is connected:
    - Let $C_1, \ldots, C_k$ be the connected components of (V, T)
    - Let $e_1, \ldots, e_k$ be the safe edge for the cuts $C_1, \ldots, \cancel{C_m}$ C_k
    - Add $e_1, \ldots, e_k$ to T

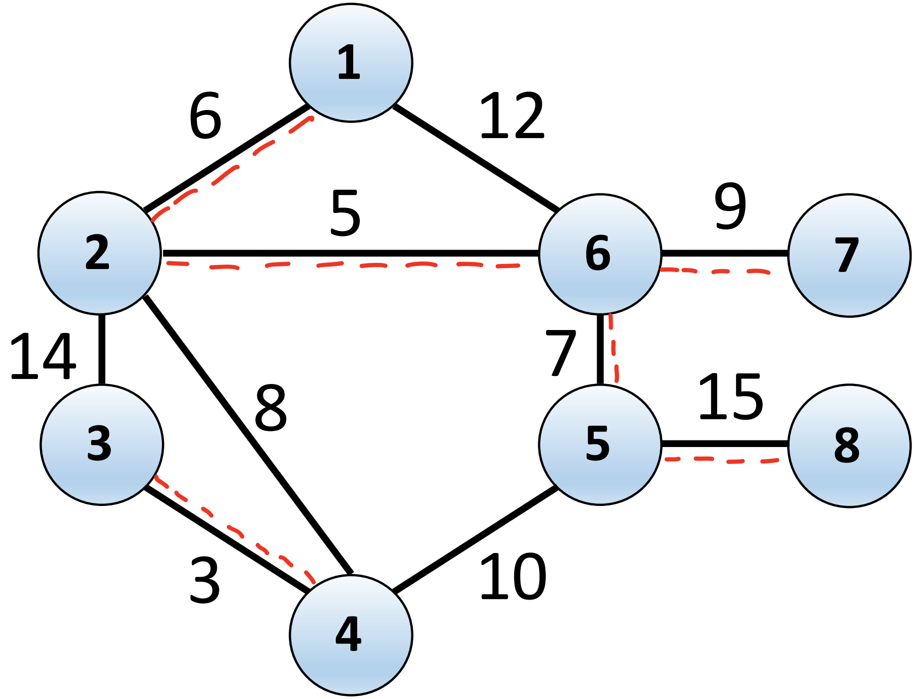
might contain duplicates

- **Correctness:** every edge we add is safe

Borůvka's Algorithm

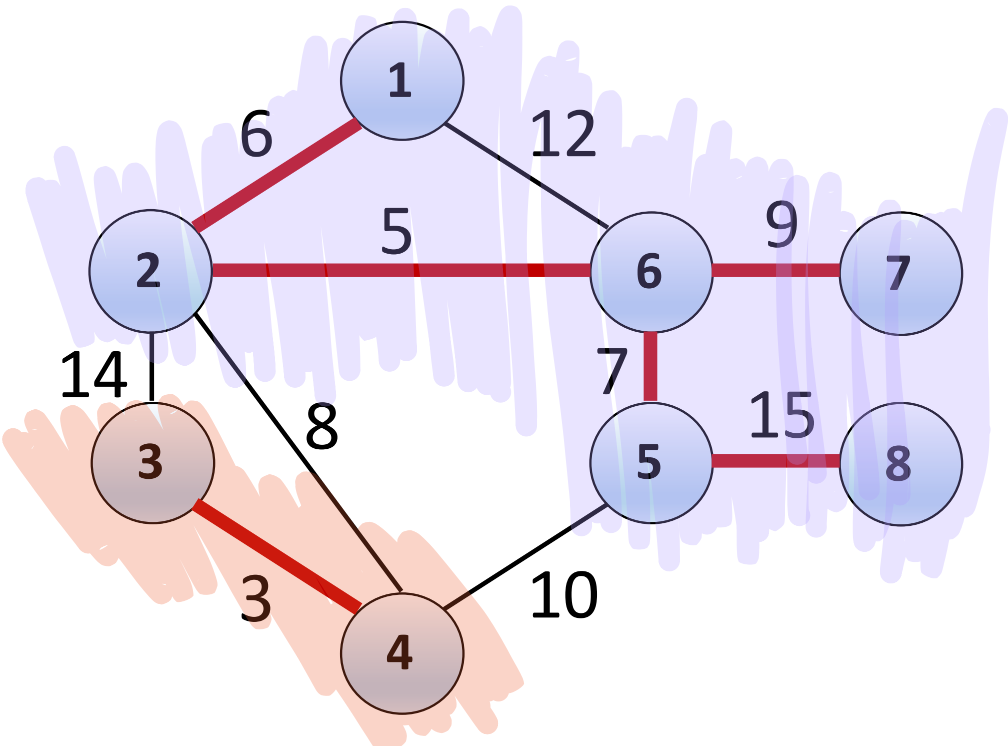
Initially $T = \emptyset$

Label Connected Components of the graph (V, T)



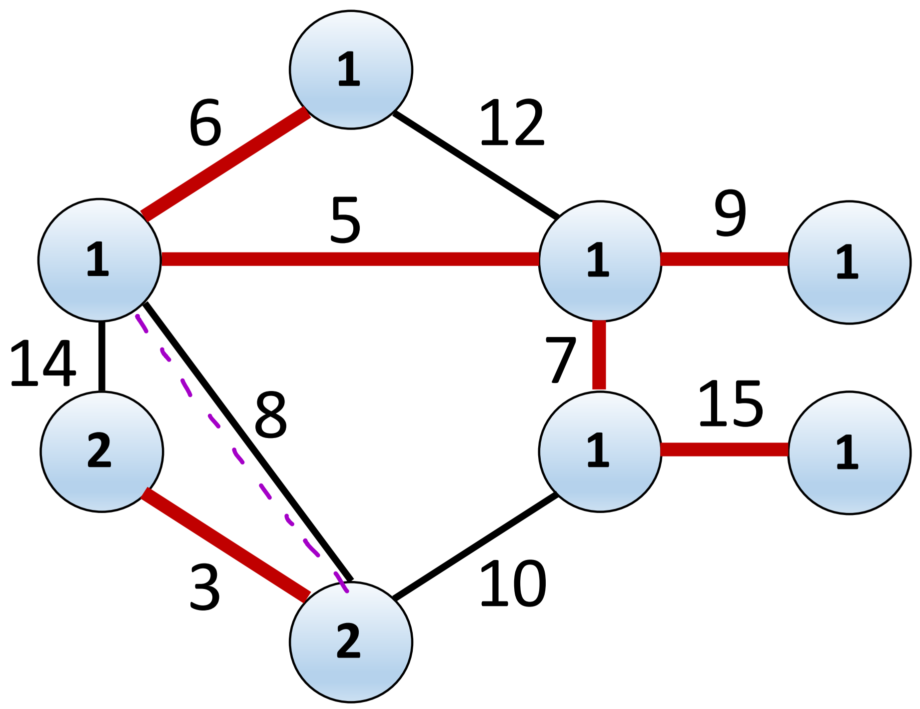
Borůvka's Algorithm

Add Safe Edges



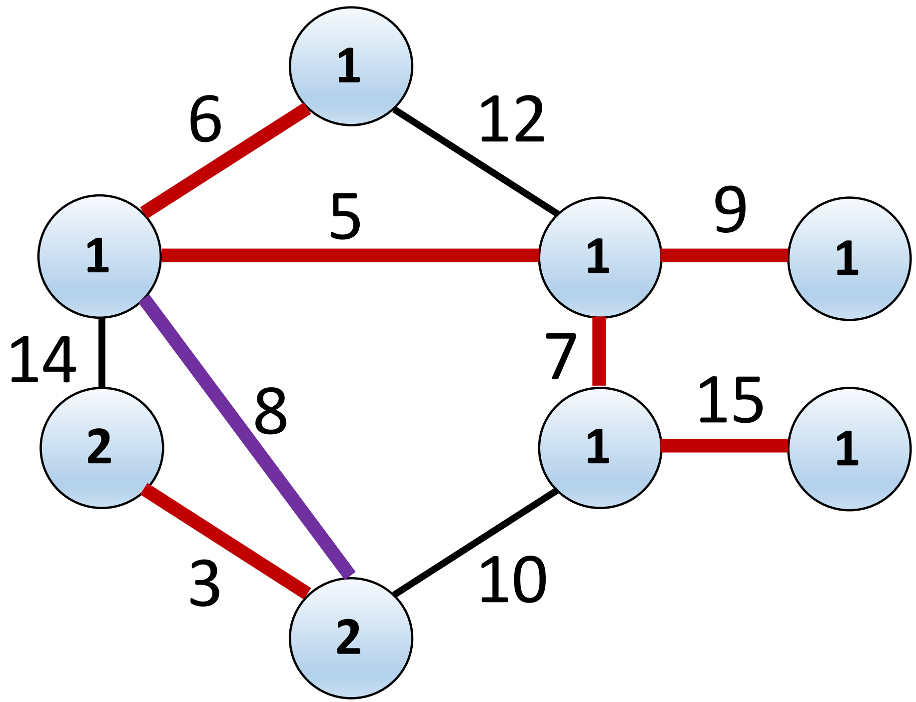
Borůvka's Algorithm

Label Connected Components



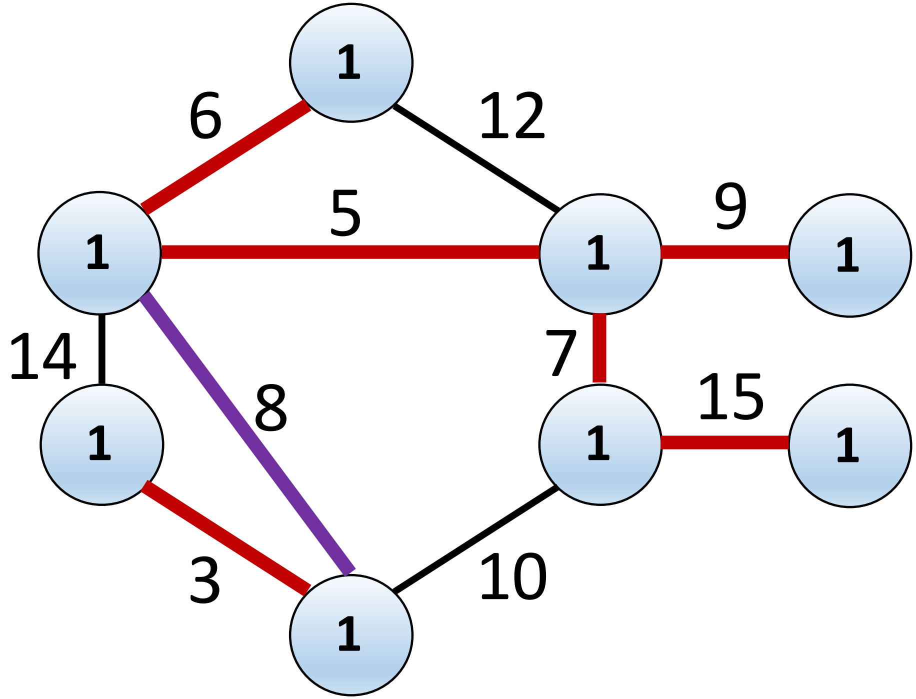
Borůvka's Algorithm

Add Safe Edges



Borůvka's Algorithm

Done!



Borůvka's Algorithm (Running Time)

- **Borůvka**
  - Let $T = \emptyset$
  - Repeat until T is connected:
    - Let $C_1, \ldots, C_k$ be the connected components of (V, T)
    - Let $e_1, \ldots, e_k$ be the safe edge for the cuts $C_1, \ldots, C_m$
    - Add $e_1, \ldots, e_k$ to T

- Running time
  - How long to find safe edges?
  - How many times through the main loop?

Borůvka's Algorithm (Running Time)

FindSafeEdges(G,T):

find connected components $C_1, \ldots, C_k$
let L[v] be the component of node v
Let S[i] be the safe edge of C_i
for each edge (u,v) in E:
    If L[u] $\neq$ L[v]:
        If w(u,v) < w(S[L[u]]):
            S[L[u]] = (u,v)
        If w(u,v) < w(S[L[v]]):
            S[L[v]] = (u,v)
Return {S[1],…,S[k]}   (Remove duplicates)

$O(m)$ time using BFS

$O(1)$ per edge

$O(m)$ total

Running Time to find safe edges is $O(m)$

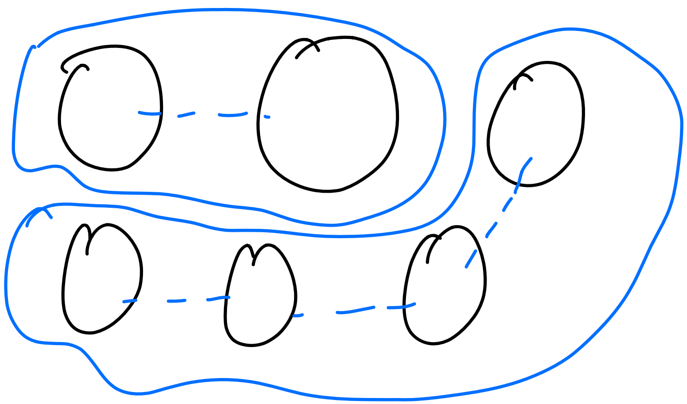
Borůvka’s Algorithm (Running Time)

- **Claim:** every iteration of the main loop halves the number of connected components.

- $\Rightarrow$ # of iterations is $O(\log n)$

- “Proof” After iteration i, we have components $C_1 \ldots C_k$

Iteration $i+1$, each component contains at least two previous components



$(\#\text{ comp's after } i+1) \leq \frac{1}{2}(\#\text{ of comp's after } i)$

Borůvka's Algorithm (Running Time)

- **Borůvka**
  - Let $T = \emptyset$
  - Repeat until T is connected:
    - Let $C_1, \dots, C_k$ be the connected components of (V, T)
    - Let $e_1, \dots, e_k$ be the safe edge for the cuts $C_1, \dots, C_m$
    - Add $e_1, \dots, e_k$ to T

- Running Time:
  - How long to find safe edges? $O(m)$
  - How many times through the main loop? $O(\log n)$

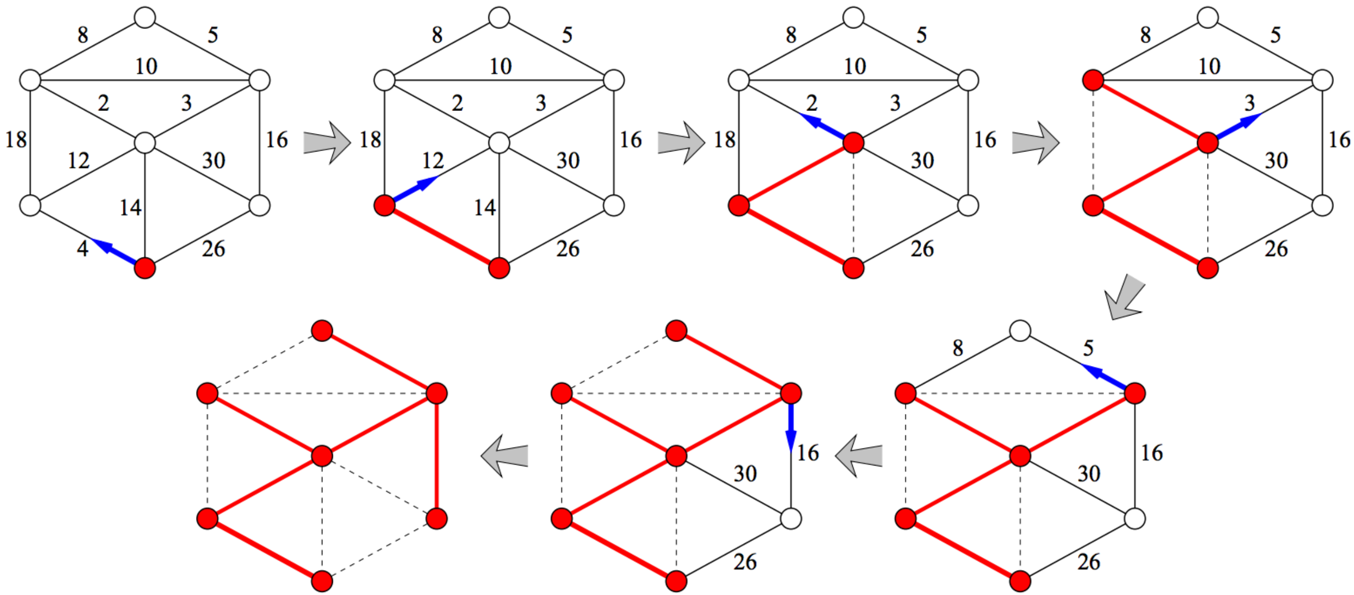
Total time: $O(m \log n)$

Prim's Algorithm

- **Prim Informal**
  - Let $T = \emptyset$
  - Let s be some arbitrary node and $S = \{s\}$
  - Repeat until $S = V$
    - Find the cheapest edge $e = (u, v)$ cut by S. Add e to T and add v to S

- **Correctness:** every edge we add is safe

Prim's Algorithm



Prim's Algorithm

```
Prim(G=(V,E))
    let Q be a priority queue storing V
        value[v] ← ∞, last[v] ←⊥
        value[s] ← 0 for some arbitrary s
    while (Q ≠ ∅):
        u ← ExtractMin(Q)
        for each edge (u,v) in E:
            if v ∈ Q and w(u,v) < value[v]:
                DecreaseKey(v,w(u,v))
                last[v] ← u

    T = {(1,last[1]),…,(n,last[n])} (excluding s)
    return T
```

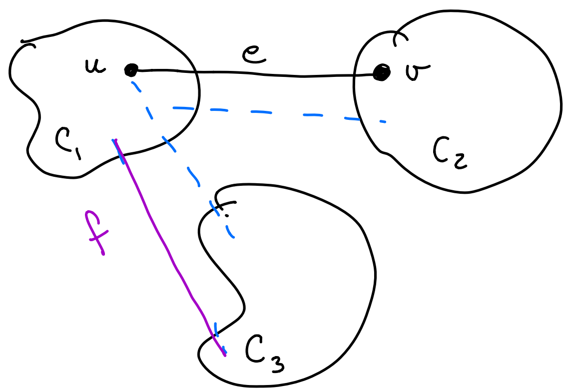
Kruskal's Algorithm

- **Kruskal's Informal**
  - Let $T = \emptyset$
  - For each edge e in ascending order of weight:
    - If adding e would decrease the number of connected components add e to T

- **Correctness:** every edge we add is safe

Claim: Every edge added by Kruskal is a safe edge.

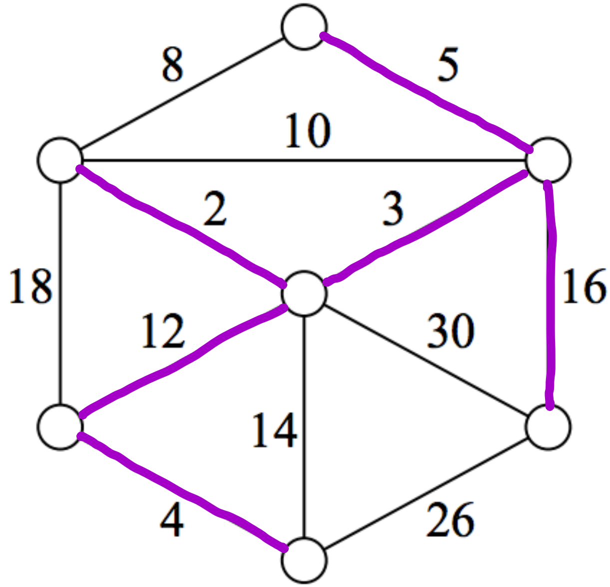
Proof: Consider some edge e, added by Kruskal, when we considered e, the T looked like



There are other edges leaving the cut C_1, suppose e were not the minimum. If $w(f) < w(e)$ then we already considered f. Why didnt we add f? At the time we considered f, its endpoints were also in two different components. But then we would have added f!

So there is no $f \in Cut(C_1)$ s.t. $w(f) < w(e)$

Kruskal’s Algorithm



Implementing Kruskal's Algorithm

- **Union-Find**: group items into components so that we can efficiently perform two operations:
  - Find(u): lookup which component contains u
  - Union(u,v): merge connected components of u,v

- Can implement **Union-Find** so that
  - Find takes $O(1)$ time
  - Any k Union operations takes $O(k \log k)$ time] Amortized running time

- Naive Implementation is an array

  Find takes O(1) time

  Union can take O(n) time

Kruskal's Algorithm (Running Time)

- **Kruskal's Informal**
  - Let $T = \emptyset$
  - For each edge e in ascending order of weight:
    - If adding e would decrease the number of connected components add e to T

- Time to sort: $O(m \log m)$
- Time to test edges: $2m$ find operations $\rightarrow$ $O(m)$ time
- Time to add edges: $n-1$ union operations $\rightarrow$ $O(n \log n)$ time

Total time is $O(m \log m)$

Implementing Union Find

(1) Maintain an array with the component of each item

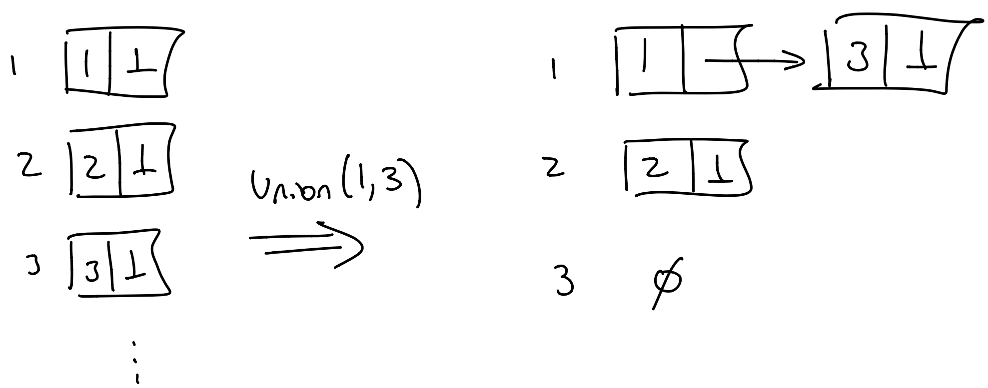
1	2	3	4	5	6	7
1	2	3	4	5	6	7

$\Downarrow$ Union(5,7)

1	2	3	4	5	6	7
1	2	3	4	5	6	5

Find = $O(1)$, Union = $O(n)$

(2) For every component, maintain a linked list of the items in that component



Union (i, j) takes time = to size of component j

(3) Keep the size of each component, merge the smaller into the bigger.

<u>Claim</u>: k unions takes $O(k \log k)$ time

<u>Pf.</u>: (1) After k unions only $O(k)$ items have changed component <u>at all</u>

(2) The largest component has size $O(k)$

(3) Every time an item changes component, the size of its component doubles.

$\Rightarrow$ no item changed component more than $O(\log k)$ times

$\therefore$ Total changes of component is $O(k \log k)$

Comparison

- **Boruvka's Algorithm:**
  - Only algorithm worth implementing
  - Low overhead, can be easily parallelized
  - Each iteration takes $O(m)$, very few iterations in practice

- **Prim's/Kruskal's Algorithms:**
  - Reveal useful structure of MSTs
  - Running time dominated by a single sort