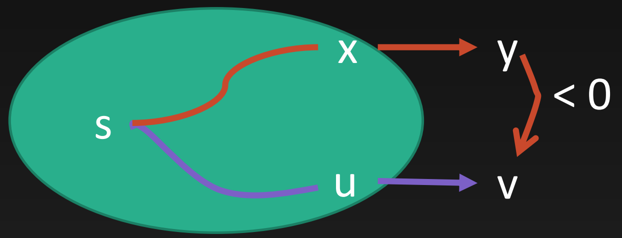
CS 4800: Algorithms & Data

Lecture 18

March 23, 2018

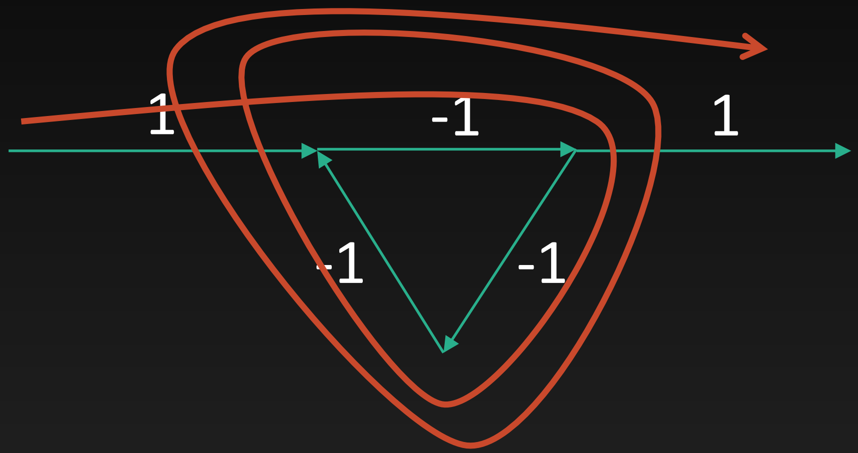
Negative weights?

- What goes wrong with previous proof?



- When v is removed, d(v) < d(y) for all unremoved y so no way shortest path goes from s to v via y

Infinitely short path?



Restrict our attention to the case with no negative cycles

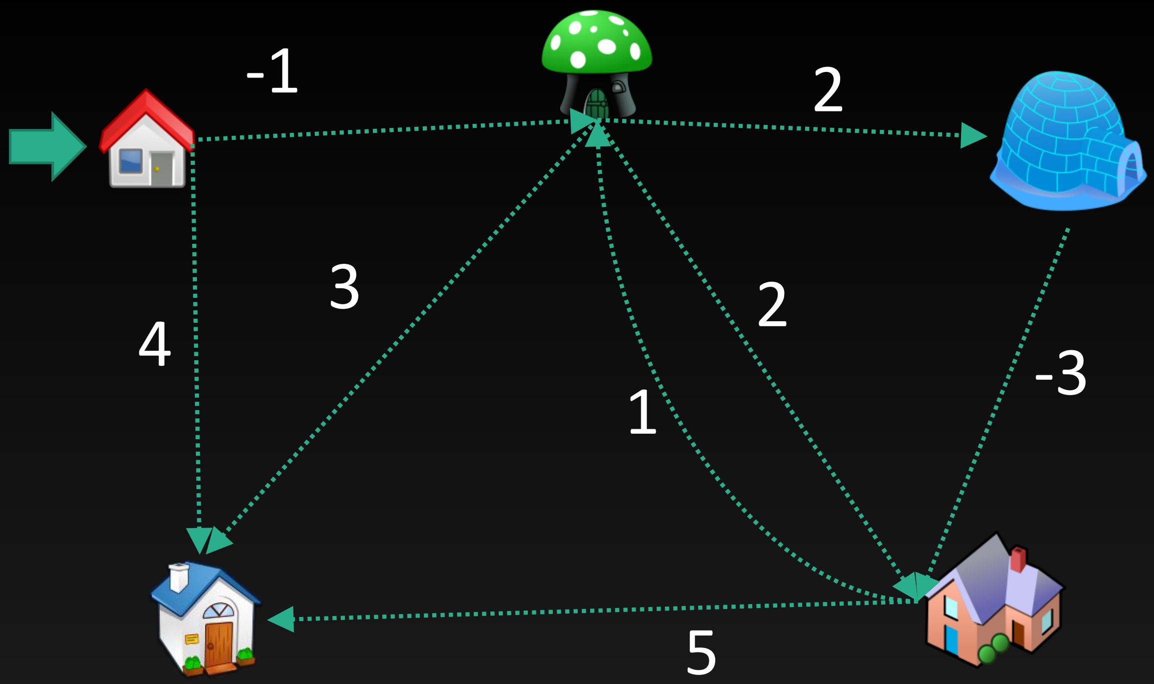
More dynamic programming

- d(i, v): min distance from s to v using at most i edges
- Without negative cycles, shortest paths use at most V-1 edges
- $d(i, v) = \begin{cases} 0 \; for \; v = s \\ \infty \; for \; v \neq s, i = 0 \\ \min_{u \in V} d(i-1, u) + w(u, v) \end{cases}$

Bellman-Ford algorithm

- Initialize $d(0, s) = 0$ and $d(0, v) = \infty$ for all $v \neq s$
- For i from 1 to V-1
  - Set $d(i, v) \leftarrow d(i-1, v)$ for all v
  - For all edges $u \rightarrow v$ in E
    - If $d(i, v) > d(i-1, u) + w(u, v)$
      - $d(i, v) \leftarrow d(i-1, u) + w(u, v)$
      - $pred(i, v) \leftarrow u$

O(VE) time



i					
0	0	∞	∞	∞	∞
1	0	4	−1	∞	∞
2	0	2	−1	1	1
3	0	2	−1	1	−2
4	0	2	−1	1	−2

All pair shortest paths

- So far, only a single source s
- What if we want shortest paths between all pairs?
  - Non-negative weights: Run Dijkstra's for all s
    - Running time: $O(V(V+E)\log V)$
  - General weights, no negative cycle: Run Bellman-Ford for all s
    - Running time: $O(V^2E)$
  - Next: better solution for general weights, no negative cycles

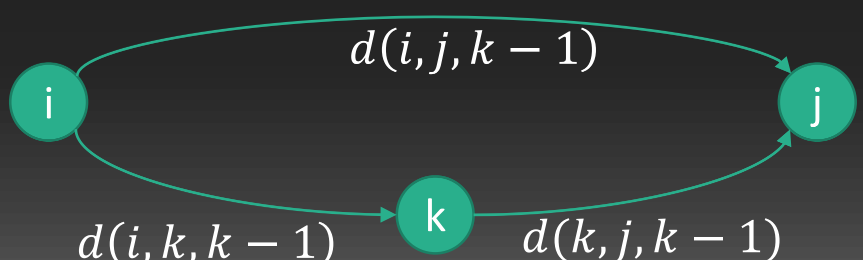
Floyd-Warshall algorithm

- d(i,j,k): Length of shortest path from i to j if we only use vertices 1…k as intermediate points
- Base cases

$$d(i,j,0) = \begin{cases} 0 \; if \; i = j \\ w(i,j) \; if \; (i,j) \in E \\ \infty \; otherwise \end{cases}$$

Recurrence relation

- How to compute d(i, j, k) from $d(*,*,k-1)$?
- Two possibilities:
  - Do not use k as an intermediate point
    $$d(i,j,k) = d(i,j,k-1)$$
  - Use k as an intermediate point
    $$d(i,j,k) = d(i,k,k-1) + d(k,j,k-1)$$
- Pick the best of two choices
  $$d(i,j,k) = \min(d(i,j,k-1), d(i,k,k-1) + d(k,j,k-1))$$



Floyd-Warshall algorithm

- Initialize $d(i,j,0) \ = \begin{cases} 0 \ if \ i=j \\ w(i,j) \ if \ (i,j) \in E \\ \infty \ otherwise \end{cases}$
- For k from 1 to V
  - For i from 1 to V
    - For j from 1 to V
      - $d(i,j,k) = \min \begin{cases} d(i,j,k-1) \\ d(i,k,k-1) + d(k,j,k-1) \end{cases}$

Save memory

- Initialize $d(i,j) = \begin{cases} 0 \; if \; i = j \\ w(i,j) \; if \; (i,j) \in E \\ \infty \; otherwise \end{cases}$
- For k from 1 to V
  - For i from 1 to V
    - For j from 1 to V
      - $d(i,j) = \min \begin{cases} d(i,j) \\ d(i,k) + d(k,j) \end{cases}$

O(V^3) time, O(V^2) space

Max flow, min cut

“Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.”

Ford-Fulkerson attributed to T. Harris

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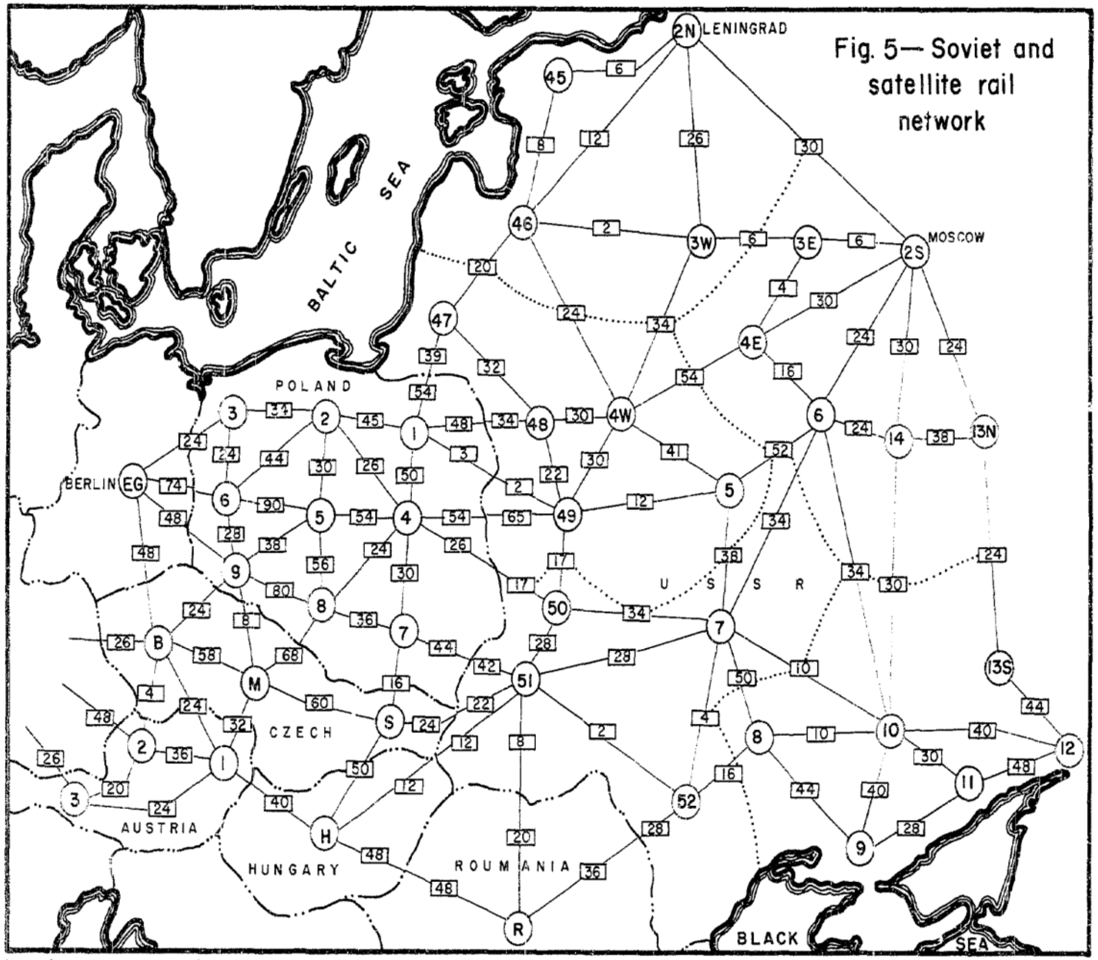

Fig. 5— Soviet and satellite rail network

Legend: —·—·— International boundary ··········· Regional boundaries of the USSR (they are included as a matter of general information)

(7) Operating divisions. Those located in Russia are believed to be accurately located. Some Russian divisions (2, 3, 4 and 13) are located in two regions and are so indicated. Divisions shown in the satellites are indicated according to the authors' best judgment, since intelligence reports are unavailable. Train capacities in Russia are for 1000-net-ton trains or their equivalent. Train capacities in Poland are for 666 net tons (or the equivalent). Train capacities in all other satellites are for 400 net tons (or the equivalent) except in East Germany. In East Germany, train capacities are those of entering lines. The numbers shown in boxes are total interdivisional capacities.

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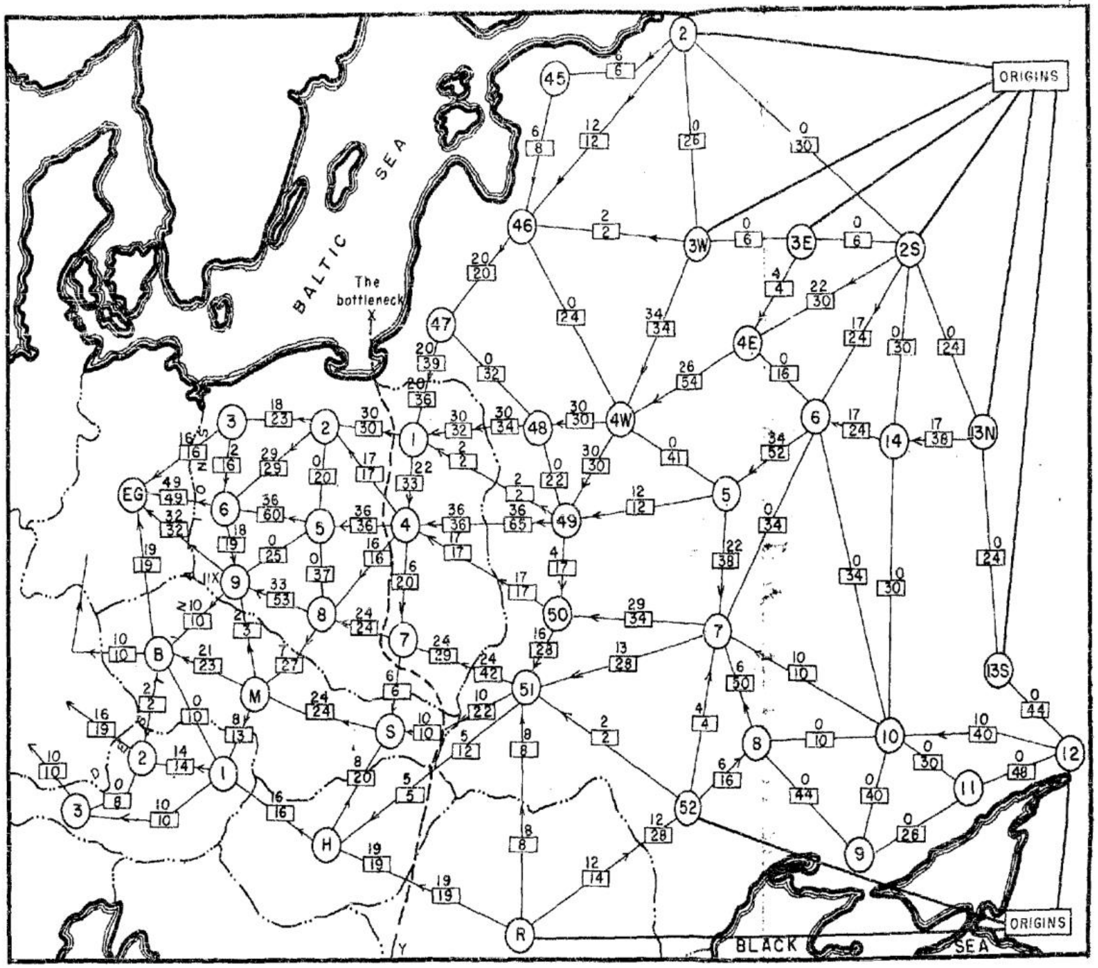

Fig. 7 — Traffic pattern: entire network available

Legend:

—··— International boundary

(8) Railway operating division

9 / [12] Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / √1000's of tons each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovavakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

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Assumption:

Entire network available for east-west traffic (no allowance for civilian or economic traffic)

Results:

(a) 163,000 tons per day can be delivered from points of origin to destinations.
(b) 147,000 tons per day can be delivered without using Austrian lines.
(c) 152,000 tons per day can be delivered into Germany by all lines.
(d) 126,000 tons per day can be delivered into East Germany without using Austrian lines.