CS 4800: Algorithms & Data

Lecture 17

March 20, 2018

Shortest paths

What is the fastest way to get from A to B?

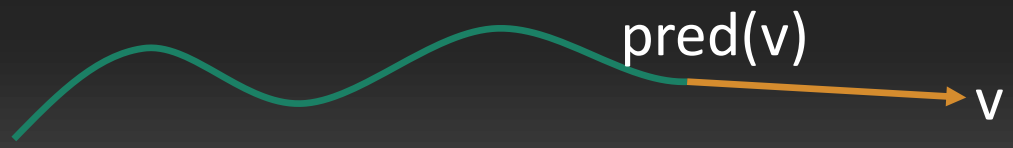
Positive edge weights

Directed graphs

Positive edge weights

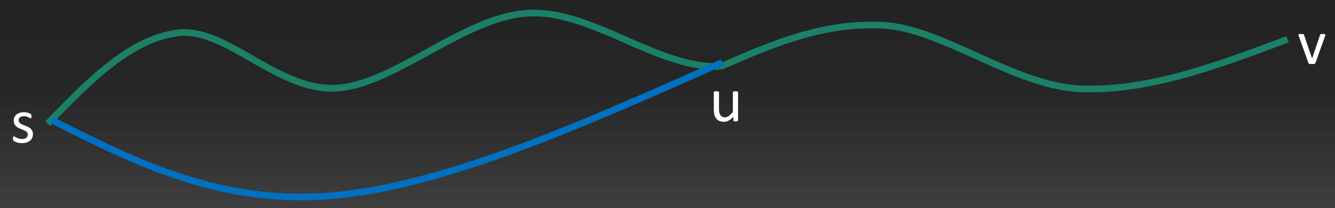
Dynamic programming

- Source vertex s
- d(v): length of shortest **tentative** path from s to v
- d*(v): length of shortest path from s to v
- pred(v): predecessor of v in shortest **tentative** path from s to v



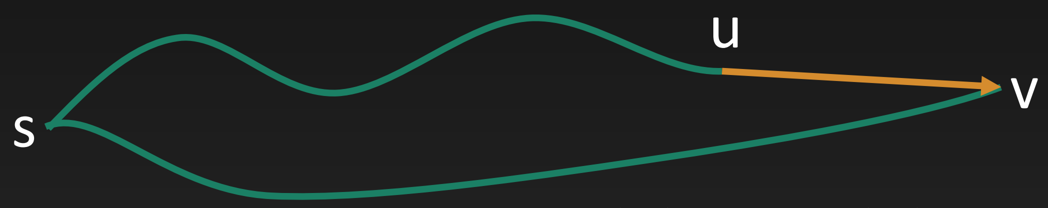
Optimal substructure

- Consider shortest path P from s to v
- Let u be a vertex on P
- The subpath of P from s to u must be shortest path from s to u
- If there is a shorter path from s to u then there is a shorter path from s to v than P



Relation among shortest distances

- Consider arbitrary edge (u,v)
- $d^*(v) \le d^*(u) + w(u, v)$
- The path $s \to u \to v$ is a feasible path from s to v



- Edge (u,v) is **tense** if $d(v) > d(u) + w(u, v)$
- When a tense edge is found, can improve $d(v)$

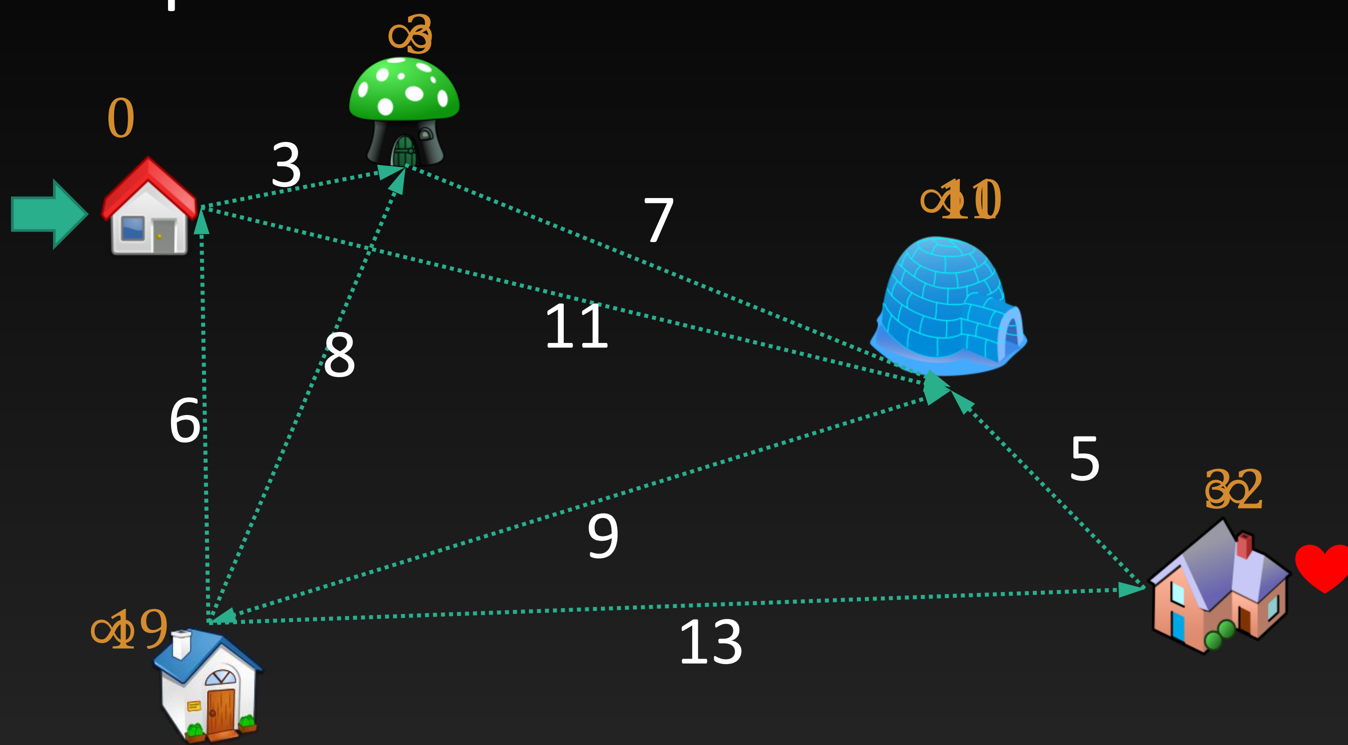
Generic shortest path algorithm

- Initialize $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$
- $Q \leftarrow \{s\}$
- While $Q \neq \emptyset$
  - Remove some u from Q
  - For all edges $u \rightarrow v$
    - If $d(v) > d(u) + w(u, v)$
      - $d(v) \leftarrow d(u) + w(u, v)$
      - $pred(v) \leftarrow u$
      - If $v \notin Q$, put v in Q. Otherwise, DecreaseKey(v).

Dijkstra's algorithm

- Initialize $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$
- $Q \leftarrow \{s\}$
- While $Q \neq \emptyset$
  - **Remove u with minimum d(u) from Q**
  - For all edges $u \rightarrow v$
    - If $d(v) > d(u) + w(u, v)$
      - $d(v) \leftarrow d(u) + w(u, v)$
      - $pred(v) \leftarrow u$
      - If $v \notin Q$, put v in Q. Otherwise, DecreaseKey(v).

Example



Q:

Correctness of Dijkstra’s

Theorem. Let S be set of nodes removed from Q. For all v in S, we have d(v)=d*(v) when v is removed from Q.

Proof. Induction over number of iterations.

First node to be removed is s and $d(s) = d^*(s) = 0.$

Assume claim is true for first k nodes.

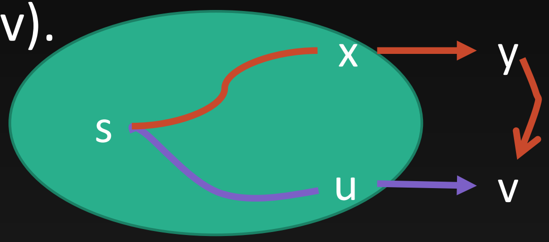
Let v be the k+1st about to be removed. Let u = pred(v).

Correctness of Dijkstra's

Let v be the k+1st about to be removed. Let u = pred(v).

u is removed from Q before (when we set pred(v) = u), so d(u) = d*(u).

Consider any other path P from s to v not via edge (u,v).



P must leave S at some point via edge (x,y).

v is about to be removed, not y, so $d(v) \leq d(y)$.

x is removed from Q so $d(x) = d^*(x)$

$d(y) \leq d(x) + w(x,y) \leq$ distance from s to y on P

Thus, $d(v) \leq Length(P)$.

Therefore, $d(v) = d^*(v)$.

Running time

- Initialize $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$
- $Q \leftarrow \{s\}$
- While $Q \neq \emptyset$

V times
  - **Remove u with minimum d(u) from Q** ← O(log V)
  - For all edges $u \rightarrow v$
    - If $d(v) > d(u) + w(u, v)$
      - $d(v) \leftarrow d(u) + w(u, v)$
      - $pred(v) \leftarrow u$
      - if $v \notin Q$, insert v into Q. Otherwise DecreaseKey(v) ← O(log V)

E times

O((V+E)log V) time

Breadth-first search

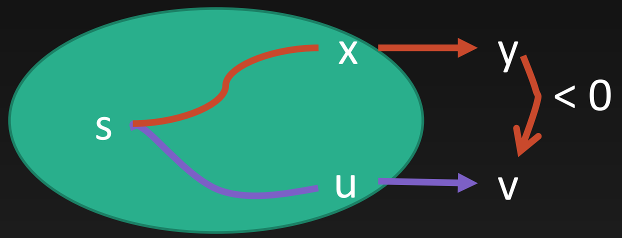
- All edge weights are 1
- Distance = #edges on the path

Breadth-first search

- Initialize $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$
- $Q \leftarrow (s,)$ ← Q is a queue: first in first out
- While $Q \neq \emptyset$
  - **Remove first u in Q**
  - For all edges $u \rightarrow v$
    - If $d(v) > d(u) + 1$
      - $d(v) \leftarrow d(u) + 1$
      - $pred(v) \leftarrow u$
      - **Put v last in Q**

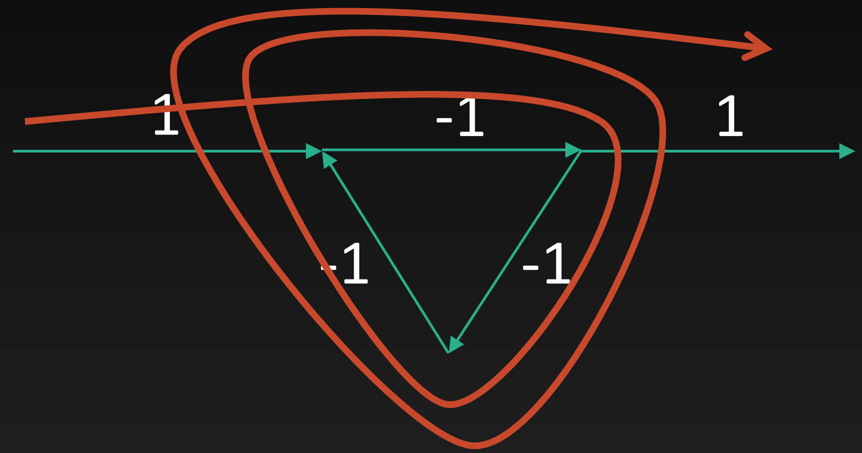
Negative weights?

- What goes wrong with previous proof?



- When v is removed, d(v) < d(y) for all unremoved y so no way shortest path goes from s to v via y

Infinitely short path?



Restrict our attention to the case with no negative cycles