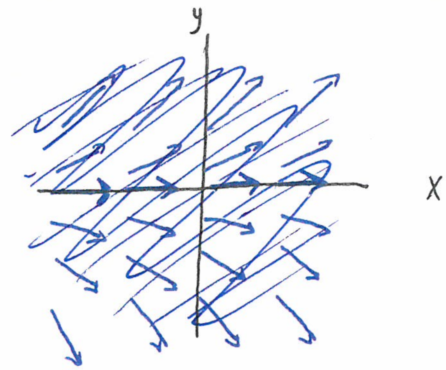
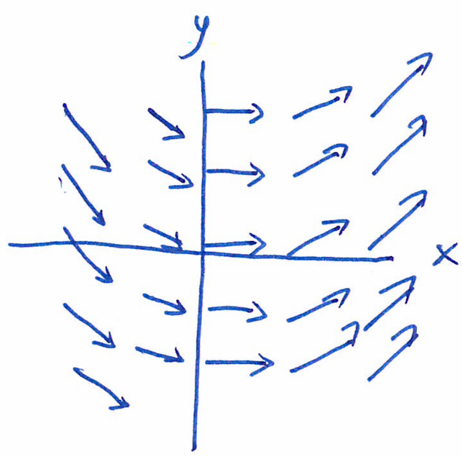
PSET 5 Solutions

1 a) $\vec{F} = \hat{\imath} + x\hat{\jmath}$



$P = 1$ $\quad \partial_y P = 0$ $\quad$ Not Conservative

$Q = x$ $\quad \partial_x Q = 1$

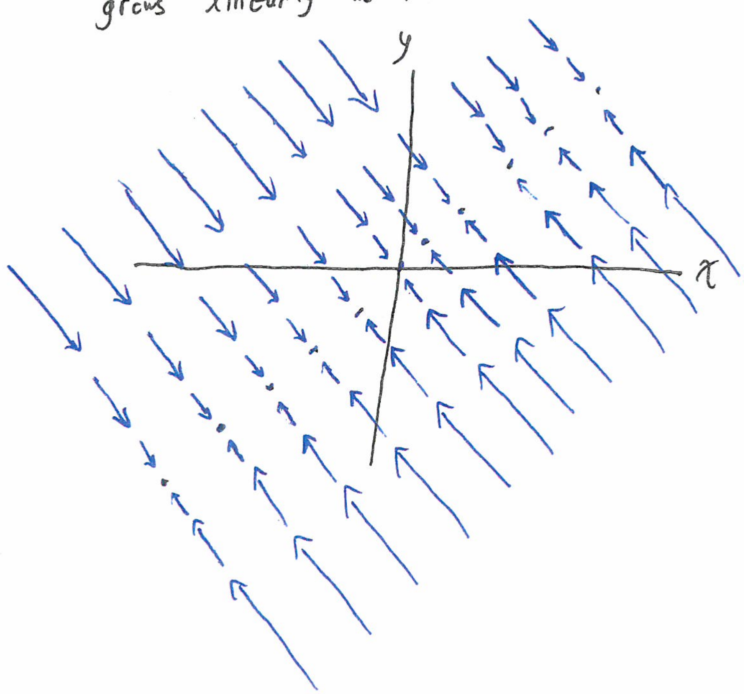


1 b) $\vec{F} = \langle y-x, x-y \rangle$

Note $\vec{F}$ is always a multiple of $\langle -1, 1 \rangle$

$\vec{F} = (x-y)\langle -1, 1 \rangle$

The multiplier $x-y$ is 0 on $y=x$ and grows linearly as x increases



$P = y-x$ $\quad \partial_y P = 1$ $\quad$ Conservative

$Q = x-y$ $\quad \partial_x Q = 1$

$$\left.\begin{array}{l} \partial_x \varphi = y-x \Rightarrow \varphi = xy - \frac{1}{2}x^2 + f(y) \\ \partial_y \varphi = x-y \Rightarrow \varphi = xy - \frac{1}{2}y^2 + g(x) \end{array}\right\} \Rightarrow \boxed{\varphi = xy - \frac{1}{2}x^2 - \frac{1}{2}y^2}$$

$\varphi = -\frac{1}{2}(x-y)^2$

1 c) $\vec{F} = \dfrac{\langle 1-y, x-2 \rangle}{\sqrt{(x-2)^2+(y-1)^2}}$

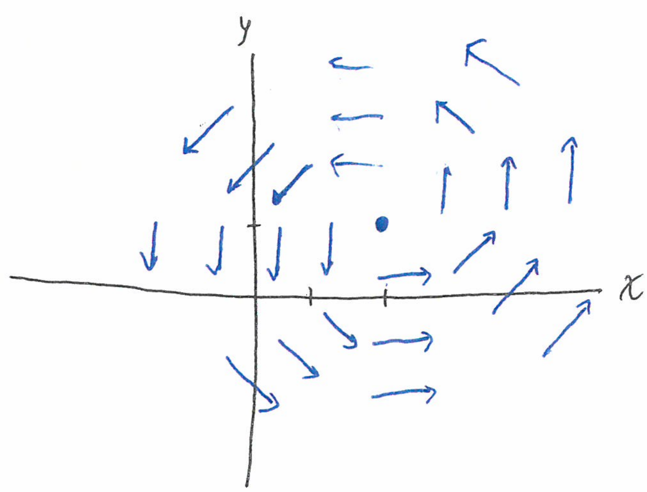
The denominator is length of numerator, so $\vec{F}$ is always a unit vector.

so lets study $\langle 1-y, x-2 \rangle$

Recall that $\langle -b, a \rangle$ is $90°$ counterclockwise from $\langle a, b \rangle$.

So $\langle 1-y, x-2 \rangle$ is $90°$ ccw rotation of $\langle x-2, y-1 \rangle$

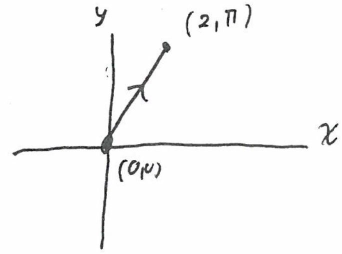
At each (x,y) direction of $\vec{F}$ is $90°$ ccw from displacement from $(2,1)$.



Not conservative because line integral $\int \vec{F} \cdot d\vec{r}$ along a positively oriented circle centered at $(1,2)$ is positive.

$$\vec{F} = \langle \cos y, -x\sin y \rangle$$

2a) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$



$\vec{r}(t) = t\langle 2, \pi \rangle$ for $0 \le t \le 1$

$\frac{d\vec{r}}{dt}(t) = \langle 2, \pi \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle \cos(\pi t), -2t\sin \pi t \rangle \cdot \langle 2, \pi \rangle\, dt$$

$$= \int_0^1 2\cos \pi t - 2\pi t \sin \pi t \quad dt$$

$$= 2\cancel{t}\cos(\pi \cancel{t}) \Big|_0^1 = -2$$

b) Identify φ st $\vec{F} = \nabla \varphi$

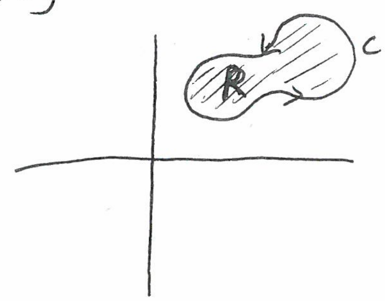
$$\left.\begin{array}{lll} \partial_x \varphi = \cos y & \Rightarrow & \varphi = x\cos y + f(y) \\ \partial_y \varphi = -x\sin y & \Rightarrow & \varphi = x\cos y + g(x) \end{array}\right\} \Rightarrow \varphi = x\cos y$$

So $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi \cdot d\vec{r} = \varphi(2,\pi) - \varphi(0,0)$

$$= -2 - 0 = -2.$$

3 a) Green's theorem: $\oint_C P\,dx + Q\,dy = \iint_R (\partial_x Q - \partial_y P)\,dA$

where R is region inside positively oriented curve C



$$\oint_C x\,dy = \iint_R (1-0)\,dA = \text{Area}(R)$$

$P = 0$ $\Rightarrow$ $\partial_y P = 0$

$Q = x$ $\quad$ $\partial_x Q = 0$

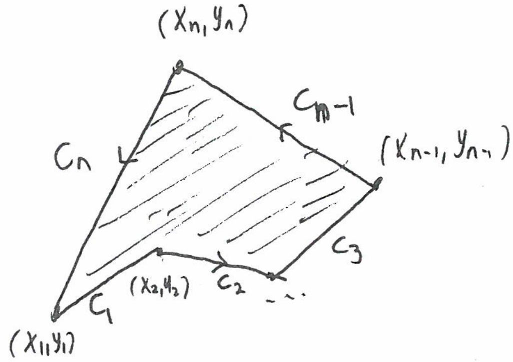
Also

$$\oint_C -y\,dx = \iint_R (0-(-1))\,dA = \text{Area}(R)$$

$P = -y$ $\Rightarrow$ $\partial_y P = -1$

$Q = 0$ $\quad$ $\partial_x Q = 0$

3b)



As per 4a, $A = \int_C x\,dy$

To evaluate the complete line integral, lets compute it over each line segment.

$$\int_C x\,dy = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n} x\,dy$$

where C_i is line segment from (x_i, y_i) to (x_{i+1}, y_{i+1})

To find $\int_{C_i} x\,dy$, write as $\int_{C_i} \langle 0, x\rangle \cdot \langle dx, dy\rangle = \int_{C_i} \langle 0, x\rangle \cdot d\vec{r}$

Parameterize: $\vec{r}(t) = \langle x_i, y_i\rangle + t\langle x_{i+1} - x_i, y_{i+1} - y_i\rangle$ for $0 \le t \le 1$

$$\frac{d\vec{r}}{dt}(t) = \langle x_{i+1} - x_i, y_{i+1} - y_i\rangle$$

so

$$\int_{C_i} x\,dy = \int_0^1 \left(x_i + t(x_{i+1} - x_i)\right)(y_{i+1} - y_i)\,dt$$

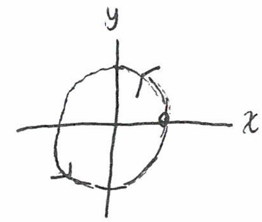
$$= x_i(y_{i+1} - y_i) + \tfrac{1}{2}(x_{i+1} - x_i)(y_{i+1} - y_i)$$

$$= \frac{x_{i+1} + x_i}{2}(y_{i+1} - y_i)$$

The area in polygon is

$$\boxed{A = \sum_{i=1}^{n} \frac{x_{i+1} + x_i}{2}(y_{i+1} - y_i) \quad \text{where } \begin{array}{l} x_{n+1} = x_1 \\ y_{n+1} = y_1 \end{array}}$$

4a) $f(r) = \frac{1}{2\pi} \int_C \phi(rx, ry)\, ds$



$$f'(r) = \frac{1}{2\pi} \int_C \left(\partial_x \phi(rx,ry)\, x + \partial_y \phi(rx,ry)\, y \right) ds$$

$$= \frac{1}{2\pi} \int_C \langle -\partial_y \phi(rx,ry), \partial_x \phi(rx,ry) \rangle \cdot \langle -y, x \rangle\, ds$$

b) $f'(r) = \frac{1}{2\pi} \int_C \langle -\partial_y \phi(rx,ry), \partial_x \phi(rx,ry) \rangle \cdot d\vec{r}$

$$= \frac{1}{2\pi} \int_C \langle -\partial_y \phi(rx,ry), \partial_x \phi(rx,ry) \rangle \cdot \langle dx, dy \rangle$$

By Green's theorem with $P = -\partial_y \phi(rx,ry)$
$Q = \partial_x \phi(rx,ry)$

$$= \frac{1}{2\pi} \iint_{\substack{\text{unit}\\ \text{disk}}} \partial_x Q - \partial_y P \; dA$$

$$= \frac{1}{2\pi} \iint \partial_{xx}\phi + \partial_{yy}\phi \; dA$$

But $\partial_{xx}\phi + \partial_{yy}\phi = 0$. So

$f'(r) = 0$. Hence f is constant

4c)

Second deriv test says

If $\phi_{xx}\phi_{yy} - \phi_{xy}^2 > 0$ and $\phi_{xx} > 0$ then local min.

If $\phi_{xx}\phi_{yy} - \phi_{xy}^2 < 0$ then saddle point

If $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0$ then inconclusive.

We are given that $\phi_{xx} + \phi_{yy} = 0$

That is, if ϕ_{xx} is nonzero, ϕ_{yy} has opposite sign. Hence ϕ would be a saddle point.

Unfortunately, the test may be inconclusive as ϕ_{xx} may be 0.

Hence, second deriv test gives some insight, but is not enough to conclude ϕ has no local minimizers.

(It does say it has no quadratic local-min behavior about a critical point).