PSET 4 solutions

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1a)

$$\int_{-\infty}^{\infty} C_1 e^{-x^2/2\sigma^2}\,dx = 1$$

So
$$\frac{1}{c} = \underbrace{\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx}_{\text{call this } I}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2\sigma^2}\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2\sigma^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2/2\sigma^2}\,r\,dr$$

$$= 2\pi\,(-\sigma^2)\,e^{-r^2/2\sigma^2}\Big|_0^{\infty}$$

$$I^2 = 2\pi\sigma^2$$

So $I = \sqrt{2\pi}\,\sigma$ $\qquad C = \dfrac{1}{\sqrt{2\pi}\,\sigma}$

$$\boxed{\phi_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2}}$$

1b) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_2\, e^{-\frac{(x^2+y^2)}{2\sigma^2}}\,dx\,dy = 1$$

So $$\frac{1}{C_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2\sigma^2}}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy$$

$$= \sqrt{2\pi}\,\sigma \quad \sqrt{2\pi}\,\sigma$$

$$= 2\pi\sigma^2$$

So $C_2 = \frac{1}{2\pi\sigma^2}$ $$\boxed{\phi_2(x,y) = \frac{1}{2\pi\sigma^2}\, e^{-\frac{(x^2+y^2)}{2\sigma^2}}}$$

1c) $$\frac{1}{C_3} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)/2\sigma^2}\,dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy \int_{-\infty}^{\infty} e^{-z^2/2\sigma^2}\,dz$$

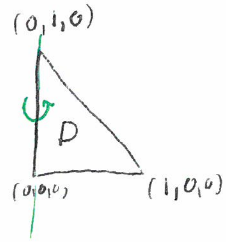
$$= \sqrt{2\pi}\,\sigma \quad \sqrt{2\pi}\,\sigma \quad \sqrt{2\pi}\,\sigma$$

$$= (2\pi)^{3/2}\sigma^3$$

$$C_3 = (2\pi)^{-3/2}\,\sigma^{-3}$$

$$\boxed{\phi_3(x,y,z) = \frac{1}{(2\pi)^{3/2}\sigma^3}\, e^{-\left(\frac{x^2+y^2+z^2}{2\sigma^2}\right)}}$$

2a)



$$I = \iint_D \rho x^2 \, dx\,dy$$

$$= \rho \int_{x=0}^{1} \int_{y=0}^{1-x} x^2 \, dy\,dx$$

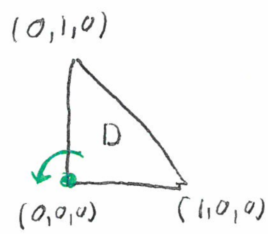
$$= \rho \int_{x=0}^{1} x^2 \left(\int_{y=0}^{1-x} dy \right) dx$$

$$= \rho \int_0^1 x^2 (1-x) \, dx$$

$$= \rho \int_0^1 x^2 - x^3 \, dx$$

$$= \rho \left[\tfrac{1}{3} x^3 \Big|_0^1 - \tfrac{1}{4} x^4 \Big|_0^1 \right.$$

$$= \boxed{\rho \left(\tfrac{1}{12} \right)}$$



2b) $I = \iint_D \rho (x^2 + y^2) \, dx\,dy$

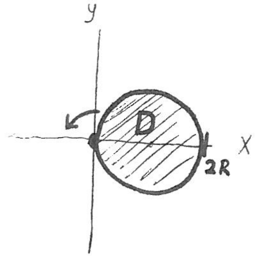
$$= \iint_D \rho x^2 dx\,dy + \iint_D \rho y^2 dx\,dy$$

Note $\iint_D \rho x^2 dx\,dy = \rho/12$

By symmetry $\iint_D \rho y^2 dx\,dy = \rho/12$

So $I = \rho/12 + \rho/12 = \boxed{\rho/6}$

2[redacted]c) The moment of inertia of D about z axis is $I = \iint_D \rho\,(x^2+y^2)\,dA$



We use polar coordinates to evaluate the integral.

First, we describe region in polar coordinates

$$-\pi/2 \leq \theta \leq \pi/2 \quad \& \quad 0 \leq r(\theta) \leq 2R\cos\theta$$

Second, we describe integrand in terms of polar

$$\rho(x^2+y^2) = \rho r^2$$

Finally, write area element in polar

$$dA = r\,dr\,d\theta$$

So
$$I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2R\cos\theta} \rho r^2\, r\,dr\,d\theta$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left(\int_0^{2R\cos\theta} r^3\,dr \right) d\theta$$

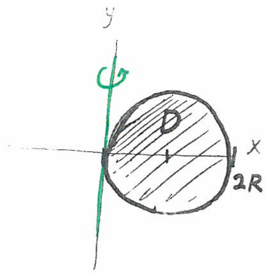
$$= \rho \int_{-\pi/2}^{\pi/2} \frac{1}{4}(2R\cos\theta)^4\,d\theta$$

$$= 4\rho R^4 \int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 4\rho R^4 \left[\frac{12x + 8\sin 2x + \sin 4x}{32} \right|_{-\pi/2}^{\pi/2}$$

$$= 4\rho R^4 \cdot \frac{3}{8}\pi = \frac{3\pi}{2}\rho R^4 = \frac{3}{2} M R^2$$

2 d)

The moment of inertia of D about y axis is



$$I = \iint_D \rho\, x^2\, dA$$

We use polar coordinates

$$I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2R\cos\theta} \rho (r\cos\theta)^2\, r\, dr\, d\theta$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left(\int_0^{2R\cos\theta} r^3 dr \right) \cos^2\theta\, d\theta$$

$$= \rho \int_{-\pi/2}^{\pi/2} \frac{1}{4} (2R\cos\theta)^4 \cos^2\theta\, d\theta$$

$$= \rho\, 2^2 R^4 \int_{-\pi/2}^{\pi/2} \cos^6\theta\, d\theta$$

$$= 4\rho R^4 \left[\frac{5x}{16} + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x \right|_{-\pi/2}^{\pi/2}$$

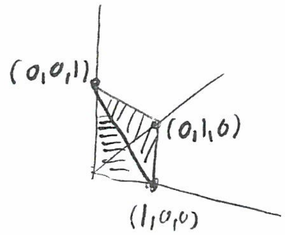
$$= 4\rho R^4 \frac{5\pi}{16}$$

$$= \frac{5}{4}\pi\rho R^4 = \frac{5}{4} M R^2$$

3) Plane going through

$(1,0,0)$ $(0,1,0)$ $(0,0,1)$

is $x+y+z=1$



$$V = \iiint_D dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1} y - yx - \tfrac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x) - (1-x)x - \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 (1-x)^2 - \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2 \, dx$$

$$= -\tfrac{1}{6}(1-x)^3 \Big|_0^1 = \tfrac{1}{6}.$$

Now, $\bar{x} = \dfrac{\iiint x\, dV}{\iiint dV}$

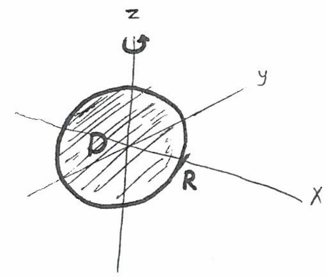
Computing, $$\iiint x\, dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x\, dz\, dy\, dx = \int_0^1 x \underbrace{\int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\, dy}_{\text{same as above}} dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2 x \, dx$$

$$= \tfrac{1}{2}\int_0^1 x - 2x^2 + x^3 \, dx = \tfrac{1}{2}\left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right] = \tfrac{1}{24}$$

Hence $\bar{x} = \dfrac{1/24}{1/6} = \dfrac{1}{4}$. By symmetry, $(\bar{x}, \bar{y}, \bar{z}) = \boxed{(1/4, 1/4, 1/4)}$

4) Consider moment of inertia about z axis.



$$I = \iiint_D \rho\,(x^2+y^2)\,dV$$

We use spherical coordinates (r, φ, θ) where φ is polar angle and θ is azimuthal angle.

First, we describe the region in spherical coordinates

$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi$$
$$0 \leq r \leq R$$

Second, we describe the integrand in terms of spherical coordinates

$$\begin{aligned} x &= r \sin\varphi \cos\theta \\ y &= r \sin\varphi \sin\theta \\ z &= r \cos\varphi \end{aligned} \quad \Rightarrow \quad x^2+y^2 = r^2 \sin^2\varphi$$

Third, we write volume element in spherical coordinates: $dV = r^2 \sin\varphi \, dr\, d\varphi\, d\theta$

$$I = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{r=0}^{R} \rho\, r^2 \sin^2\varphi \; r^2 \sin\varphi \; dr\, d\varphi\, d\theta$$

$$= \rho \int_{r=0}^{R} r^4\, dr \int_{\theta=0}^{2\pi} d\theta \int_{\varphi=0}^{\pi} \sin^3\varphi \, d\varphi$$

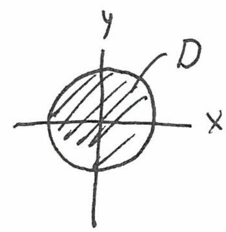
$$= \rho \, \tfrac{1}{5} R^5 \cdot 2\pi \left[\frac{\cos 3\varphi}{12} - \frac{3}{4}\cos\varphi \right]_0^{\pi}$$

$$= \tfrac{2}{5} \rho \pi R^5 \left[4/3 \right]$$

$$= \tfrac{2}{5} \left(\tfrac{4}{3} \pi R^3 \right) R^2 = \boxed{\tfrac{2}{5} M R^2}$$

5 a) $$\bar{d} = \frac{\int_{-R}^{R} x\,dx}{\int_{-R}^{R} 1\,dx} = \frac{\frac{1}{2}x^2\Big|_{-R}^{R}}{2R} = \frac{R^2}{2R} = \boxed{\tfrac{1}{2}R}$$

b) $$\bar{d} = \frac{\iint_D \sqrt{x^2+y^2}\,dx\,dy}{\iint_D dx\,dy}$$



Note: $\iint_D dx\,dy = \pi R^2$

Convert to polar

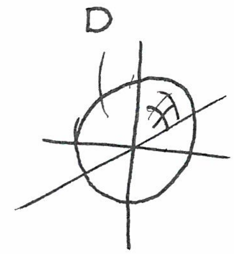
$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r\; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \int_0^R r^2\,dr$$

$$= 2\pi\;\frac{1}{3}R^3 = \frac{2}{3}\pi R^3$$

So $$\bar{d} = \frac{\frac{2}{3}\pi R^3}{\pi R^2} = \boxed{\frac{2}{3}R}$$

5c) $$\bar{d} = \frac{\iiint_D \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz}{\iiint_D dx\,dy\,dz}$$



Note $\iiint_D dx\,dy\,dz = \text{Volume of } D = \frac{4}{3}\pi R^3$

Use spherical coordinates

$$\iiint_D \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi}\int_{r=0}^{R} r\, r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^R r^3\, dr$$

$$= 2\pi \underbrace{\left(-\cos\varphi\Big|_0^{\pi}\right)}_{2} \frac{1}{4}R^4$$

$$= \frac{4\pi}{4}R^4 = \pi R^4$$

So $$\bar{d} = \frac{\pi R^4}{\frac{4}{3}\pi R^3} = \boxed{\frac{3}{4}R}$$

d) In 1d, $\bar{d} = \frac{1}{2}R$

In 2d, $\bar{d} = \frac{2}{3}R$

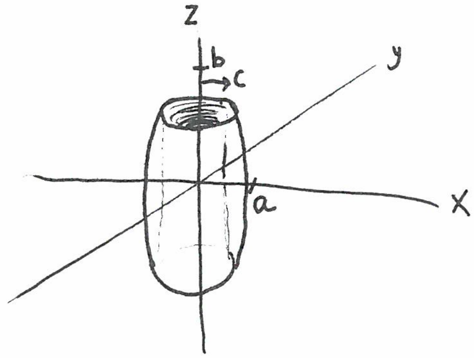
In 3d, $\bar{d} = \frac{3}{4}R$

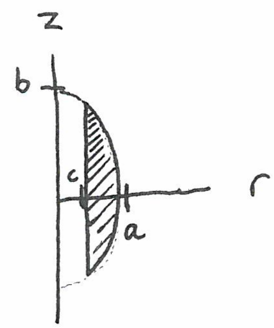
In nd, we suspect $\bar{d} = \frac{n}{n+1}R$

That is, for large dimensions, the average distance to the origin within a hypersphere of radius R is $\approx R$. So almost all of the sphere's volume is at its edge! Quite surprising!

6)

a)





b) Use cylindrical coordinates

First we describe the shape in cylindrical coordinates

Note azimuthal symmetry, $0 \leq \theta \leq 2\pi$

Note $c \leq r \leq a$

To find z for a fixed r, lets write formula for ellipsoid in cylindrical coords

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{is} \quad \frac{r^2}{a^2} + \frac{z^2}{b^2} = 1$$

So $z = \pm b\sqrt{1 - r^2/a^2}$

$$V = \int_{\theta=0}^{2\pi} \int_{r=c}^{a} \int_{-b\sqrt{1-r^2/a^2}}^{b\sqrt{1-r^2/a^2}} 1\, r\, d\theta\, dr\, dz$$

$$= \int_0^{2\pi} d\theta \int_{r=c}^{a} 2b\sqrt{1-r^2/a^2}\, r\, dr = 4\pi b \int_{r=c}^{a} \sqrt{1-r^2/a^2}\, r\, dr$$

$$= 4\pi b \left[-\frac{a^2}{3}\left(1 - r^2/a^2\right)^{3/2} \right|_c^a = \boxed{\frac{4}{3}\pi b\,(a^2 - c^2)}$$