21 July 2014
Calculus 3, Interphase 2014
Paul E. Hand
hand@math.mit.edu

# Problem Set 4

Due: **28 July 2014** in class.

1. (15 points) A probability distribution is a nonnegative function that integrates to 1.
   (a) In 1d, $\phi_1(x) = c_1 \exp(-\frac{x^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of $c_1$ such that $\int_{-\infty}^{\infty} \phi_1(x)dx = 1$.
   (b) In 2d, $\phi_2(x, y) = c_2 \exp(-\frac{x^2+y^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of $c_2$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_2(x, y)dxdy = 1$.
   (c) In 3d, $\phi_3(x, y, z) = c_3 \exp(-\frac{x^2+y^2+z^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of $c_3$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_3(x, y, z)dxdydz = 1$.
2. (20 points) Moments of inertia for planar objects.
   (a) Consider the triangular shape between $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$. Suppose it has constant density $\rho$. What is its moment of inertia about the $y$ axis?
   (b) What is its moment of inertia about the $z$ axis?
   (c) Consider the disk of radius $R$, lying in the $xy$ plane. It's center is at $(R, 0, 0)$, and it has constant density $\rho$. What is it's moment of inertia about the $z$ axis? Use polar coordinates. In polar coordinates, this circle is given by $r(\theta) = 2R \cos\theta$ for $-\pi/2 \leq \theta \leq \pi/2$.
   (d) What is its moment of inertia about the $y$ axis? Use polar coordinates.
3. (15 points) Find the volume and center of mass of the tetrahedron between $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Assume the tetrahedron has constant density.
4. (10 points) Find the moment of inertia of a constant-density sphere rotating about any axis containing its center. Express the answer in terms of $cMR^2$, where $M$ is the sphere's mass.
5. (15 points) *The Curse of Dimensionality*
   (a) Find the average value of the distance to the origin for the 1d points $-R \leq x \leq R$.
   (b) Find the average value of the distance to the origin for the 2d disk of radius $R$.
   (c) Find the average value of the distance to the origin for the 3d ball of radius $R$.
   (d) What do you think is the corresponding value for a ball in $n$ dimensions? Does this surprise you?
6. (15 points) *The Olive Problem.* An olive can be formed by taking the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

   and removing the part that is inside the cylinder $x^2 + y^2 = c^2$. Assume $c < a$.
   (a) Make a 3d sketch of the olive and label it with $a$, $b$, and $c$. Make a $rz$ sketch of the olive, as per cylindrical coordinates.
   (b) Compute the volume of the olive.