PSET 1 Solutions

$\vec{a} = (1,1,1)$ $\qquad$ $\vec{b} = (-1,2,2)$

1 a) Recall $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

We now compute $\vec{a}\cdot\vec{b}$, $|\vec{a}|$, $|\vec{b}|$

$$\vec{a}\cdot\vec{b} = 1(-1)+1(2)+1(2) = 3$$

$$|\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(-1)^2+2^2+2^2} = \sqrt{9} = 3$$

So $\cos\theta = \dfrac{3}{\sqrt{3}\,3} = \dfrac{1}{\sqrt{3}}$. $\boxed{\theta = \cos^{-1}\frac{1}{\sqrt{3}}}$

b) Area of parallelogram spanned by $\vec{a}$ & $\vec{b}$ is $|\vec{a}\times\vec{b}|$

Computing cross product,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= \vec{i}\,(1\cdot2-1\cdot2) - \vec{j}\,(1\cdot2-1(-1)) + \vec{k}\,(1\cdot2-1(-1))$$

$$= 0\,\vec{i} - 3\vec{j} - 3\vec{k}$$

The length $|\vec{a}\times\vec{b}|$ is $\sqrt{0^2+(-3)^2+(-3)^2} = \boxed{3\sqrt{2}}$

c) Recall, the plane containing $\vec{a}$ with normal vector $\vec{b}$ is

$$\vec{b} \cdot \vec{x} = \vec{b} \cdot \vec{a}$$

Here, $\vec{a} = (1,1,1)$ $\quad \vec{b} = (-1,2,2)$ $\quad \vec{x} = (x,y,z)$

We compute $\vec{b} \cdot \vec{a} = (-1)\cdot 1 + 2(1) + 2\cdot 1 = 3$

$$\vec{b} \cdot \vec{x} = -x + 2y + 2z$$

So plane is $\boxed{(-1,2,2) \cdot \vec{x} = 3 \quad \text{or} \quad -x + 2y + 2z = 3}$

d) To specify a plane, we need a point on plane $\vec{x}_0$ and a normal vector, $\vec{n}$. The plane is then $\vec{x} \cdot \vec{n} = \vec{x}_0 \cdot \vec{n}$

We choose $\vec{x}_0 = (0,0,0)$ as the point.

To find a normal vector $\vec{n}$, we use the information that the plane contains $\vec{a}$ and $\vec{b}$.

Because the normal vector is perpendicular to all vectors along the plane,

$$\vec{n} \cdot (\vec{a} - \vec{0}) = 0$$

$$\vec{n} \cdot (\vec{b} - \vec{0}) = 0$$

Careful. If $\vec{0}$ is not in plane, $\vec{n}$ would be perp to $\vec{a} - \vec{c}$ & $\vec{b} - \vec{c}$

So $\vec{n}$ is perpendicular to $\vec{a}$ & $\vec{b}$.

Cross product gives such a vector

$\vec{n}$ can be taken to be $\vec{a} \times \vec{b} = (0,-3,-3)$

So, plane containing $\vec{0}, \vec{a}, \vec{b}$ is $(0,-3,-3) \cdot \vec{x} = 0$ or $\boxed{(0,1,1) \cdot \vec{x} = 0 \text{ aka } y + z = 0}$

aka $-3y - 3z = 0$

Let $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$

2 a) To show $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, we will compute the cross product $\vec{a} \times \vec{b}$, then the dot product with a.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i}\,(a_2 b_3 - a_3 b_2) - \vec{j}\,(a_1 b_3 - a_3 b_1) + \vec{k}\,(a_1 b_2 - a_2 b_1)$$

$$= (a_2 b_3 - a_3 b_2,\ -a_1 b_3 + a_3 b_1,\ a_1 b_2 - a_2 b_1)$$

Now we compute $\vec{a} \cdot (\vec{a} \times \vec{b})$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = a_1 (a_2 b_3 - a_3 b_2) + a_2 (-a_1 b_3 + a_3 b_1) + a_3 (a_1 b_2 - a_2 b_1)$$

$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} - \cancel{a_1 a_2 b_3} + \cancel{a_2 a_3 b_1} + \cancel{a_1 a_3 b_2} - \cancel{a_2 a_3 b_1}$$

$$= 0.$$

2b) Let $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

To show $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$, we will explicitly compute $\vec{a} \times \vec{b}$ & $\vec{b} \times \vec{a}$.

From before, $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1)$

We compute $\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ in the same manner

$$\vec{b} \times \vec{a} = \vec{i}(b_2 a_3 - b_3 a_2) - \vec{j}(b_1 a_3 - b_3 a_1) + \vec{k}(b_1 a_2 - a_1 b_2)$$

$$= \cancel{(b_2 a_3 - b_3 a_2)}$$

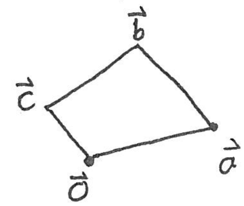
$$= (b_2 a_3 - b_3 a_2, -b_1 a_3 + b_3 a_1, b_1 a_2 - a_1 b_2)$$

We observe $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

3 a)

Step 1/2: Describe Quadrilateral with as few variables as possible.

Let D be the origin. The quadrilateral is given by three vectors $\vec{a}, \vec{b}, \vec{c}$:



Step 3: Interpret knowns & goal in terms of vectors

Knowns: Midpoint of $\vec{0}$ to $\vec{b}$ Equals midpoint of $\vec{c}$ to $\vec{a}$

That is: $\frac{\vec{b}+\vec{0}}{2} = \frac{\vec{a}+\vec{c}}{2}$

Goal: Want to show that line segment from $\vec{a}$ to $\vec{b}$ is same length & direction as line segment $\vec{0}$ to $\vec{c}$.

That is, want to show $\vec{b}-\vec{a} = \vec{c}-\vec{0}$

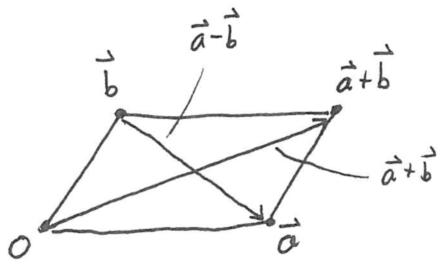
Step 4: Algebra to complete proof.

We know $\frac{\vec{b}+\vec{0}}{2} = \frac{\vec{a}+\vec{c}}{2}$.

Thus $\vec{b} = \vec{a}+\vec{c}$,

and $\vec{b}-\vec{a} = \vec{c}-\vec{0}$.

3b)



Step 1 & 2: Describe with as few variables as possible.

Make one corner the origin. A parallelogram is given by two vectors $\vec{a}$ & $\vec{b}$.

The diagonals as drawn are $\vec{a}+\vec{b}$ & $\vec{a}-\vec{b}$

Step 3: Interpret knowns & goal in terms of vectors

Known: $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Goal: $\vec{a}\cdot\vec{b}=0$.

Step 4: Algebra to complete proof.

Given $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$.

Hence, $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$\Rightarrow \cancel{\vec{a}\cdot\vec{a}} + 2\vec{a}\cdot\vec{b} + \cancel{\vec{b}\cdot\vec{b}} = \cancel{\vec{a}\cdot\vec{a}} - 2\vec{a}\cdot\vec{b} + \cancel{\vec{b}\cdot\vec{b}}$

$\Rightarrow 4\vec{a}\cdot\vec{b}=0$

$\Rightarrow \vec{a}\cdot\vec{b}=0$,

So $\vec{a}\perp\vec{b}$ & the parallelogram is a rectangle.

4)

No, $-2y-2z=1$ is a plane, not a line.

This plane contains the line of intersection between $x+y+z=1$ and $x-y-z=2$.

5 a)

A plane is given by

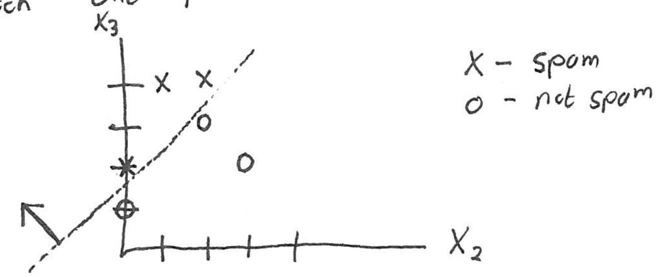
$$\vec{n} \cdot \vec{x} = b.$$

We need to find $\vec{n}$ & b such that

$\vec{n} \cdot \vec{x} > b$ for all spam $\vec{x}$

$\vec{n} \cdot \vec{x} < b$ for all nonspam $\vec{x}$.

We sketch the points in the $x_2 x_3$ plane



Observe we can separate x & o with dashed line with slope 1 and x_3-intercept 1.5. ~~That is a normal vector to~~

Viewing this line as a plane that is x_1-invariant we identify that the plane contains $\begin{pmatrix} 0 \\ 0 \\ 1.5 \end{pmatrix}$

and has normal vector $\vec{n} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Our plane is given by $(0,-1,1) \cdot \vec{x} = (0,-1,1) \cdot (0,0,1.5) = 1.5$.

The separating plane is $\boxed{(0,-1,1) \cdot \vec{x} = 1.5}$

To verify, we compute $\vec{n} \cdot \vec{x}$ for all spam & non spam.

Spam:

$(0,-1,1) \cdot (2,0,2) = 2 > 1.5$

$(0,-1,1) \cdot (3,1,4) = 3 > 1.5$

$(0,-1,1) \cdot (1,2,4) = 2 > 1.5$

Not spam:

$(0,-1,1) \cdot (2,0,1) = 1 < 1.5$

$(0,-1,1) \cdot (0,2,3) = 1 < 1.5$

$(0,-1,1) \cdot (1,3,2) = -1 < 1.5$

We have successfully found a plane separating spam & non spam.

b) From (a), we know

$$\vec{n} \cdot \vec{x} > b \quad \text{for sample spam } \vec{x}$$

$$\vec{n} \cdot \vec{x} < b \quad \text{for sample non spam } \vec{x}.$$

w/ $\vec{n} = (0, -1, 1)$ & $b = 1.5$.

To classify new data $\vec{x}$, we compute $\vec{n} \cdot \vec{x}$ & compare with $\vec{b}$

i) $(0,-1,1) \cdot (2,1,1) = 0 < 1.5.$ Not spam.

ii) $(0,-1,1) \cdot (0,0,3) = 3 > 1.5$ spam.

Note, answers may or may not differ to 6b if a different hyperplane in 6a was found.