Lecture 8

7/21/2014

Double Integrals in Cartesian

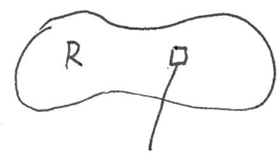
— — in Polar

Warm up

2d shape R. Bacteria has ~~density~~ concentration $C(x,y)$, $\frac{\text{bacteria}}{\text{cm}^2}$

Write double integral for total # bacteria.

$$\iint_R C(x,y)\, dA$$

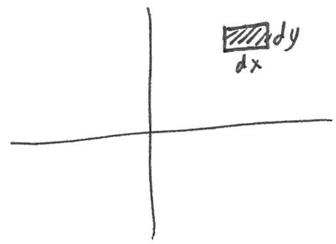


small square at (x,y) has $C(x,y)\Delta A$ bacteria

Area element in cartesian

A region of size dx & dy has aran

$dA = dxdy$



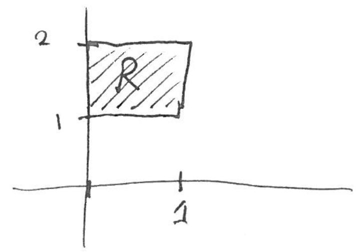
Computing double integrals over rectangles

If $R = [a,b] \times [c,d]$, compute

$$\iint_R f(x,y)\,dA \quad \text{by iterated integral}$$

- $\displaystyle\int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx$ OR
- $\displaystyle\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy$

Example: Find $\displaystyle\iint_{[0,1]\times[1,2]} xy\,dA$



- Sketch region of integration
- Write as iterated integral

$$\int_{x=0}^{1}\Bigg(\underbrace{\int_{y=1}^{2} xy\,dy}_{\substack{x \text{ is constant, so} \\ \text{pull it outside}}}\Bigg)dx$$

$$x\int_{y=1}^{2} y\,dy = x\,\tfrac{1}{2}y^2\Big|_1^2 = x\left(2-\tfrac{1}{2}\right) = \tfrac{3}{2}x$$

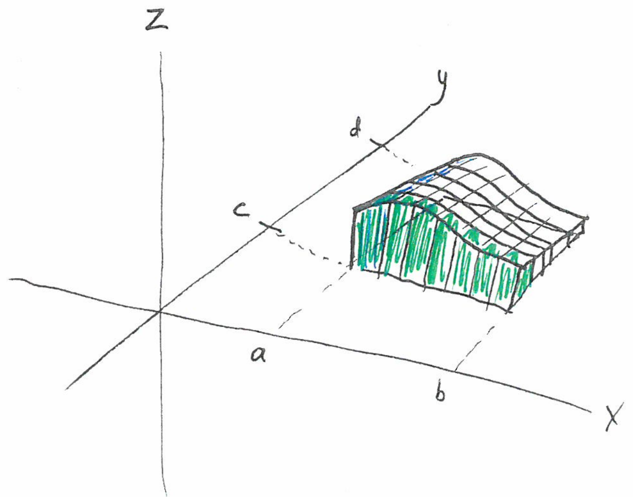
$$= \int_{x=0}^{1} \tfrac{3}{2}x\,dx = \tfrac{3}{2}\cdot\tfrac{1}{2}x^2\Big|_0^1 = \boxed{\tfrac{3}{4}}$$

Double Integrals as Iterated integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\, \overset{dA}{\cancel{dx\,dy}} = \int_{x=a}^{b} \Bigg(\underbrace{\int_{y=c}^{d} f(x,y)\, dy}_{\substack{x \text{ is constant.} \\ \text{This is 1d integral whose value depends on } x}} \Bigg) dx = \int_{y=c}^{d} \Bigg(\int_{x=a}^{b} f(x,y)\, dx \Bigg) dy$$

integrand has no y dependence (it was integrated away), so this is function of x only.
1d integral

Why?



Partition into rectangles of size $\Delta x\, \Delta y$

Select one row of these columns. (green)

Volume of this part $\approx \int_a^b f(x,c)\, dx \cdot \Delta y$

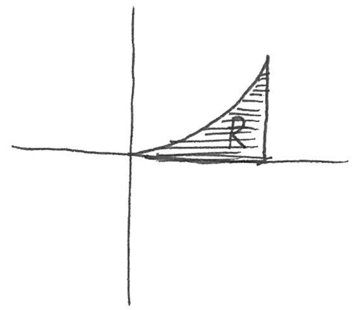
Row corresponding to y has volume $\approx \int_a^b f(x,y)\, dx \;\; \Delta y$

Adding up these gives over all y's $\quad \sum \left(\int_a^b f(x,y_i)\, dx \right) \Delta y \approx \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$

Example: Specify region between $y=0$, $y=x^2$, $x=1$

a) as an x-dependent range of y values

b) as a y-dependent range of x values



a) $x_{min} = 0$ at x, $y_{min}(x) = 0$

$x_{max} = 1$ $y_{max}(x) = x^2$

Region is $0 \leq x \leq 1$ $0 \leq y \leq x^2$

b) $y_{min} = 0$ at y $x_{min}(y) = \sqrt{y}$

$y_{max} = 1$ $x_{max}(y) = 1$

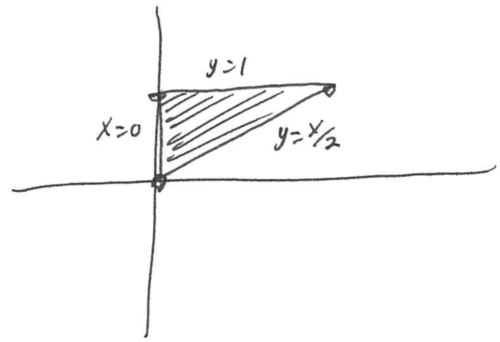
Region is $0 \leq y \leq 1$ $\sqrt{y} \leq x \leq 1$

Activity

Specify triangle bounded by (0,0) (2,1) (0,1) as

a) X-dependent range in y

b) y-dependent range of X

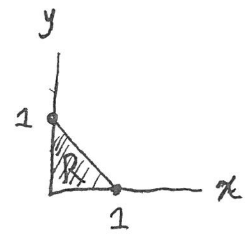


a) $0 \leq x \leq 2$

$x/2 \leq y \leq 1$

b) $0 \leq y \leq 1$

$0 \leq x \leq 2y$

Example: $I = \iint_R xy \, dA$ with 

Describe region:

x ranges from 0 to 1

At fixed x, y ranges from 0 to $1-x$

OR

y ranges from 0 to 1

At fixed y, x ranges from 0 to $1-y$

Write as iterated integral

$$I = \int_{x=0}^{1} \left(\int_{y=0}^{1-x} xy \, dy \right) dx$$

$$= \int_{x=0}^{1} x \left(\int_{y=0}^{1-x} y \, dy \right) dx$$

b/c x is constant in inner integral

$$= \int_0^1 x \; \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - 2x^2 + x) \, dx$$

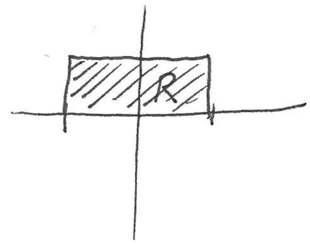
$$= \frac{1}{2} \left[\tfrac{1}{4} x^4 \Big|_0^1 - \tfrac{2}{3} x^3 \Big|_0^1 + \tfrac{1}{2} x^2 \Big|_0^1 \right]$$

$$= \frac{1}{24}$$

Compute:

Activity:

$$I = \iint_{[-1,1]\times[0,1]} x y^2 \, dA$$



$$I = \int_{x=-1}^{1} \int_{y=0}^{1} x y^2 \, dy \, dx$$

$$I = \int_{x=-1}^{1} x \left(\int_{y=0}^{1} y^2 dy \right) dx$$

Inner integral $\displaystyle\int_{y=0}^{1} y^2 dy = \frac{1}{3}$

$$I = \int_{x=-1}^{1} x/3 \, dx = \frac{1}{3} \frac{1}{2} x^2 \Big|_{-1}^{1} = 0.$$

Must be 0 because integrand is odd.

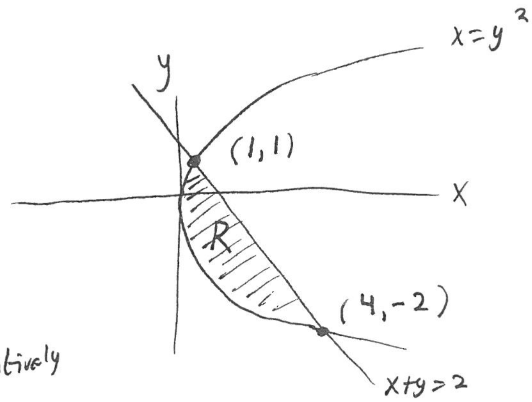
Example:

Set up iterated integral for

$I = \iint_R xy^2 \, dA$ where R is region

between curves $x = y^2$ & $x + y = 2$

(1) Draw region:



(2) Describe region Quantitatively

Complicated to describe as $y_{max}(x)$ & $y_{min}(x)$.

Easy to describe as $x_{max}(y)$ $x_{min}(y)$

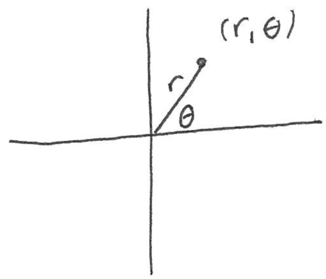
y varies from -2 to 1

At fixed y, x varies from y^2 to $2-y$

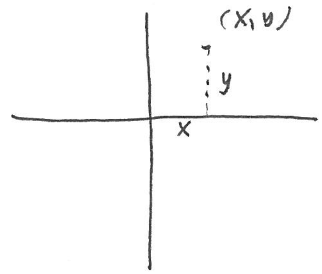
$$I = \int_{y=-2}^{1} \left(\int_{y^2}^{2-y} xy^2 \, dx \right) dy$$

which could be solved.

Polar Coordinates



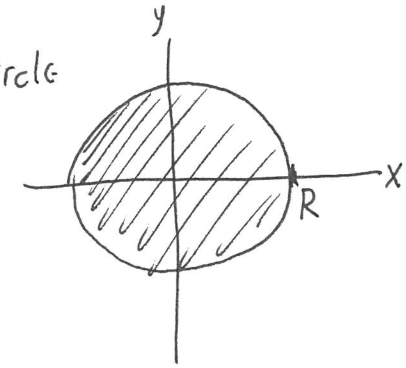
Cartesian Coordinates



$x = r\cos\theta$
$y = r\sin\theta$

Specify a region in polar coordinates

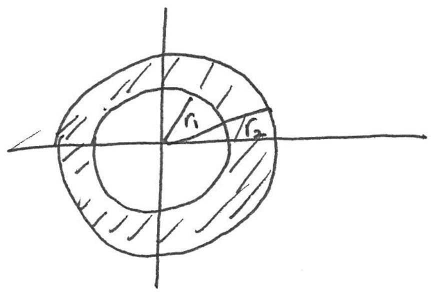
Circle



$0 \le r \le R$

$0 \le \theta < 2\pi$

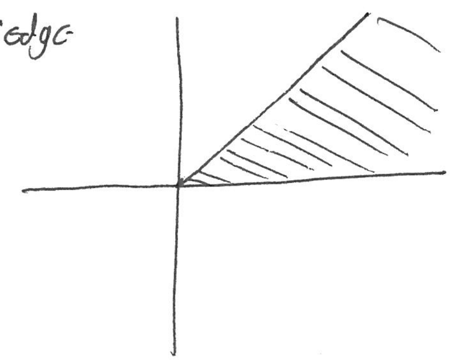
Annulus



$r_1 \le r \le r_2$

$0 \le \theta < 2\pi$

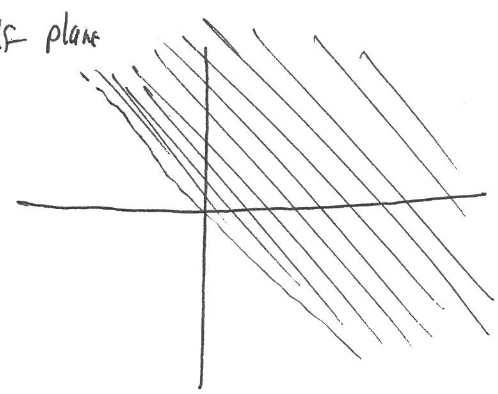
Wedge



$0 \le r < \infty$

$0 \le \theta \le \pi/4$

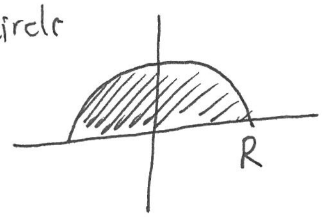
Half plane



$0 \le r < \infty$

$-\frac{\pi}{4} \le \theta \le 3\pi/4$

Semicircle



$0 \le r \le R$

$0 \le \theta \le \pi$