Lecture 7

7/15/2014

Lagrange Multipliers

Double Integrals

Game

~~Activity~~: Consider a smooth $f(x,y)$ w/ $f(0,0)=0$
$\nabla f(0,0)=0$

That is, $f(x,y) \approx ax^2 + bxy + cx^2$

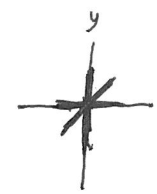
a) Suppose f is concave up in $\hat{i}$ direction
& — — — — — $\hat{j}$ direction

Is $(0,0)$ a local minimizer of f?

No. $f(x,y) = x^2 - 3xy + y^2$ $\quad H = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ $\quad \det H = -5 < 0$

Saddle

b) Suppose f is also concave up in $\hat{i}+\hat{j}$ direction



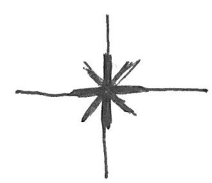
Is $(0,0)$ a local min?

No $f(x,y) = x^2 + 3xy + y^2$ $\quad H = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ $\quad \det H = -5 < 0$

Saddle

Observe $f(\varepsilon,\varepsilon) = 5\varepsilon^2$
$f(-\varepsilon,\varepsilon) = -\varepsilon^2$

c) Suppose f is further concave up in $-\hat{i}+\hat{j}$ direction



No $f(x,y) = x^2 + 5xy + 5y^2$

$f(\varepsilon,\varepsilon) = 11\varepsilon^2$ but $\quad H = \begin{pmatrix} 1 & 5 \\ 5 & 5 \end{pmatrix}$ $\quad \det H = -20 < 0$

$f(-\varepsilon,\varepsilon) = \varepsilon^2$

Saddle

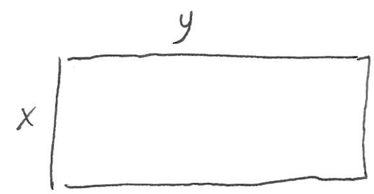
Warmup: $g(x,y) = x^2 + 3xy + y^2$

$f(x,y) = x^2 + 5xy + 5y^2$

For both f & g,
Is $(0,0)$ a min, max, or saddle?

Example:

Find rectangle w/ largest area given perimeter P



max xy s.t. $2x+2y-P=0$

Write Lagrangian

$$L(x,y,\lambda) = xy - \lambda(2x+2y-P)$$

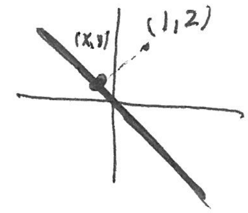
Set all partials of L to 0

$$\left.\begin{aligned} \partial_x L &= y - 2\lambda = 0 \\ \partial_y L &= x - 2\lambda = 0 \end{aligned}\right\} \Rightarrow x = y \Rightarrow \text{square!}$$

$$\partial_\lambda L = (2x+2y-P) = 0$$

Activity:

Find nearest point on line $x+y=0$ to $(1,2)$ using Lagrange multipliers



a) Write as $\min f(x,y)$ subject to $g(x,y)=0$

$$\min\ (x-1)^2+(y-2)^2 \quad \text{s.t.} \quad x+y=0$$

b) Write $L(x,y,\lambda)$

$$L(x,y,\lambda) = (x-1)^2+(y-2)^2 - \lambda(x+y)$$

c) Set partials of L to 0

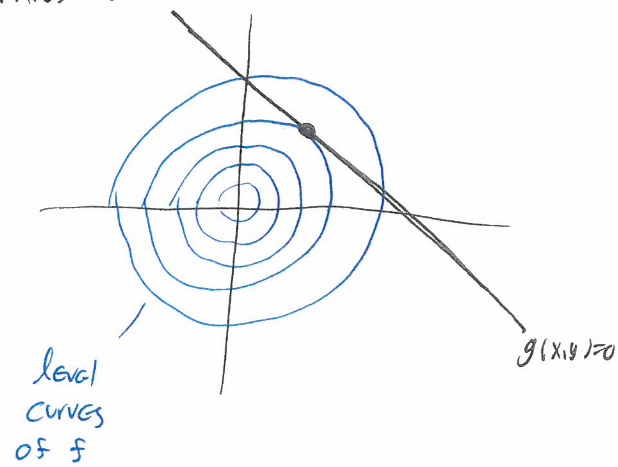
$$\partial_x L = 2(x-1) - \lambda = 0 \qquad \Rightarrow \quad 2(x-1) = 2(y-2)$$

$$\partial_y L = 2(y-2) - \lambda = 0 \qquad \Rightarrow \quad y = x+1$$

$$\partial_\lambda L = -(x+y) = 0 \qquad \text{But } x=-y \quad \text{so} \quad \boxed{\begin{array}{l} y = 1/2 \\ x = -1/2 \end{array}}$$

Justification of Lagrange Multipliers

max/min $f(x,y)$ st $g(x,y)=0$



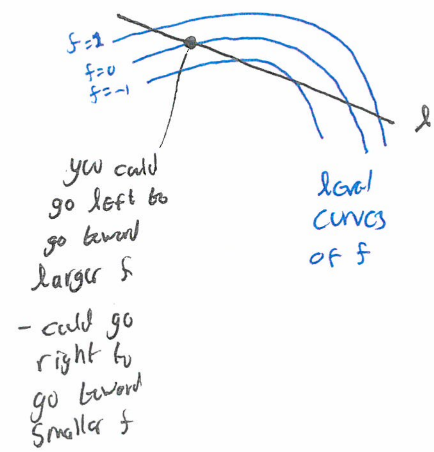
$\mathcal{L} = f(x,y) + \lambda g(x,y)$

$\nabla \mathcal{L} = 0 \Rightarrow \nabla f(x,y) + \lambda \nabla g(x,y) = 0$

∇f is parallel to ∇g at constrained extremum

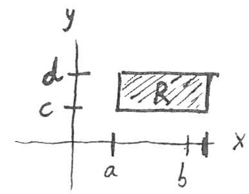
So level curves of f & g are tangent.

If level curves of f & g weren't tangent, you could move along constraint and increase OR decrease objective

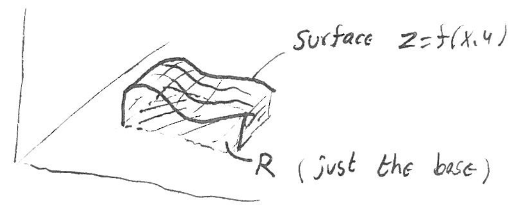


Double Integrals and Volume

Let $R = [a,b] \times [c,d]$ be rectangle



Consider surface $z = f(x,y)$.



Volume under surface & over R

is $\displaystyle\iint_R f(x,y)\,dA$

double integral over R

small bit of area

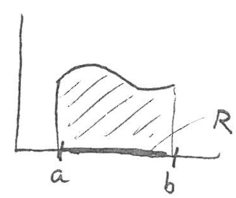
Analogy to 1d:

Consider curve $y = f(x)$

Let $R = [a,b]$.

Area under curve & over R

is $\displaystyle\int_R f(x)\,dx$

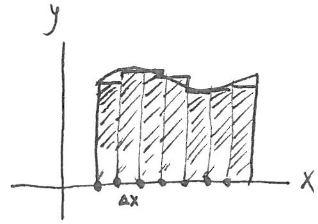


Caution: If f is negative, it counts as negative area/volume.

Riemann Sums

Riemann Sums in 1d

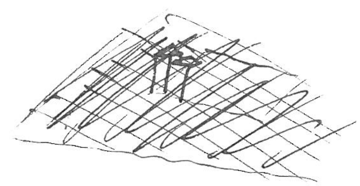
$$\int_{[a,b]} f(x)\,dx \approx \sum f(x_i)\Delta x$$

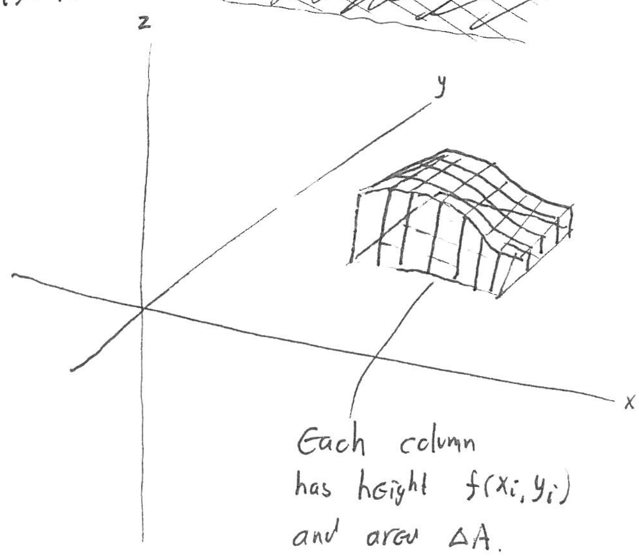


Break region into small strips, add up area of each strip.

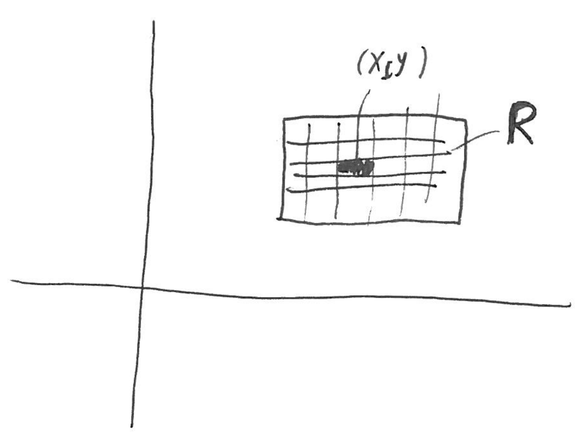
Riemann Sums in 2d

- $$\iint_{[a,b]\times[c,d]} f(x,y)\,dA \approx \sum f(x_i,y_i)\Delta A$$

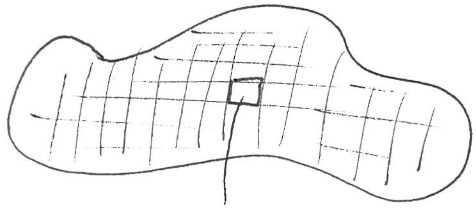




- As $\Delta A \to 0$, get better approximation of overall volume



Area of region R

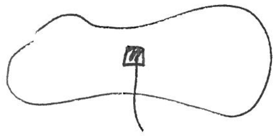


area dA. contribution to total area is $1\, dA$

Area of $R = \iint_R 1\, dA$

Mass of 2d plate with area density $\rho(x,y)$

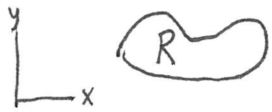
$\rho(x,y)$ has units kg/m^2



area dA Contributes to total mass $\rho(x,y)\, dA$

$M = \iint_R \rho(x,y)\, dA$

Activity: Suppose plate has area density $\rho(x,y)$



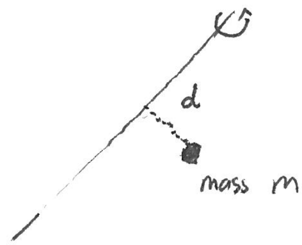
Put it on a fulcrum at position $x = x_0$

Recall: torque due to force F at a distance d from fulcrum is $F \cdot d$.

Write double integral for total torque.

$$\iint_R \rho(x,y)\,(x-x_0)\,g\,dA$$

Moment of inertia



A mass M rotating about an axis with distance d to axis has moment of inertia md^2.

For a little square area ΔA at (x,y)

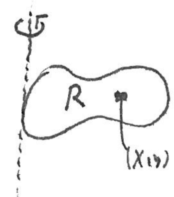
mass = $\rho(x,y)\,\Delta A$

distance to axis = $d(x,y)$

MoI contribution $\rho(x,y)\,d^2(x,y)\,\Delta A$

$$I = \iint_R \rho(x,y)\,d^2(x,y)\,dA.$$

Example:



object rotating about y axis.

$d(x,y) = |x|$

$d^2(x,y) = x^2$

$$I = \iint_R \rho(x,y)\,x^2\,dA$$

Activity:

a) Write down dable integral for 2d plate R rotatd about z axis.

b) Write down dable integral for 2d plate R rotatd about $y = 2$.