Lecture 5 - Gradient, directional derivatives, chain rule - 7/11/2014 — Interphase 2014 Calc 3

17. Gradient and directional derivative

a. The gradient of $f(x, y)$ at (x, y) is $\overrightarrow{\nabla} f(x, y) = \langle \partial_x f(x, y), \partial_y f(x, y) \rangle$.

The gradient of $f(x, y, z)$ at (x, y, z) is $\overrightarrow{\nabla} f(x, y, z) = \langle \partial_x f(x, y, z), \partial_y f(x, y, z), \partial_z f(x, y, z) \rangle$.

b. First order approximation: $f(\vec{x} + \overrightarrow{\Delta x}) = f(\vec{x}) + \overrightarrow{\nabla} f(\vec{x}) \cdot \overrightarrow{\Delta x}$

c. The directional derivative of $f(\vec{x})$ in the direction $\vec{u}$ (with $|\vec{u}| = 1$) at $\vec{x}$ is

$$D_{\vec{u}} f(\vec{x}) = \lim_{\epsilon \to 0} \frac{f(\vec{x} + \epsilon \vec{u}) - f(\vec{x})}{\epsilon} .$$

Directional derivative is a scalar.

d. Relation of directional derivative to gradient:

$$D_{\vec{u}} f(\vec{x}) = \overrightarrow{\nabla} f \cdot \vec{u} \text{ when } |\vec{u}| = 1$$

e. The gradient of a function at a point is perpendicular to the level set containing that point.

The gradient of a function at a point is a vector in the direction of steepest ascent of that function at that point.

The gradient of a function at a point is a vector with magnitude equal to the directional derivative in the direction of steepest ascent.

f. The normal vector to a surface can be found by

- viewing the surface as a level set of a function of three variables
- taking the gradient of that function

18. Chain rule and total derivative

a. Multivariable chain rule:

$$\partial_a f(x(a, b), y(a, b)) = \partial_x f \partial_a x + \partial_y f \partial_a y$$

b. Total derivative:

$$\frac{d}{dt} f\left(t, x(t), y(t), z(t)\right) = \partial_t f + \partial_x f \frac{dx}{dt} + \partial_y f \frac{dy}{dt} + \partial_z f \frac{dz}{dt}$$