Lecture 4.5 10 july 2013

Parameterizations of complex motions

Velocity, Speed, Accelerabion, Tangent vector, Arc Length

Quadratic Surfaces

Parameterizations of complex motions

To parameterize a curve consisting of multiple parts (translation + rotation + ...) :

(1) • Describe ~~Each~~ desired point as vector sum of each component

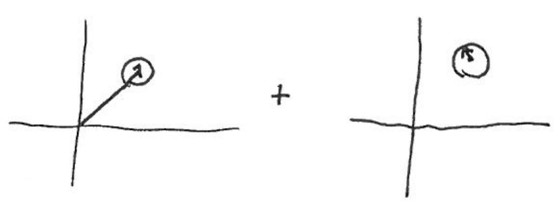
(2) • Identify convenient parameter. when in doubt use time

(3) • Identify each components dependence on parameters

Example: Frisbee of radius r initially centered at $(0,0)$. Translates in direction $(1,1)$ with speed V. Rotates c.c.w. at ω rad/sec. Point originally at $(r,0)$ is painted. what curve is traced out?

(1) Sum of translation + rotation

$$\vec{X} = \vec{X}_{center} + \vec{X}_{rot}$$



(2) Given rates w.r.t. time, so use t

(3) $\vec{X}_{centr}(t) = (1,1)\,t$? No. Didnt use V.

what is vector in dir of $(1,1)$ w/ length vt ?

$$\vec{X}_{centr}(t) = \underbrace{\frac{(1,1)}{|(1,1)|}}_{\text{dir }(1,1)} v t = \cancel{\frac{(Vt, Vt)}{\sqrt{2}}}\ \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) vt$$

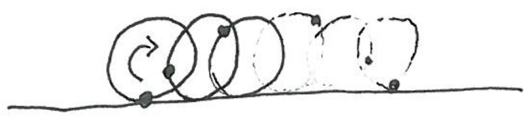
$\vec{X}_{rotat}(t) = (r\cos wt, r\sin wt)$ Note, checks at at $t=0$

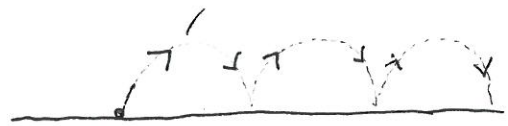
Combining

$$\boxed{\vec{X}(t) = \left(\frac{1}{\sqrt{2}}vt + r\cos wt, \frac{1}{\sqrt{2}}vt + r\sin wt\right) \quad -\infty < t < \infty}$$

Example: Cycloid.

Cylinder (radius r) rolls along flat ground at angular speed ω. Initially point touching ground is painted. As it rolls, what path is traced out?

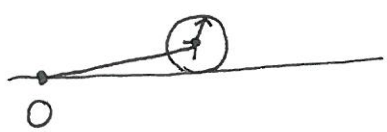




1) Break motion into separate parts:

center of cylinder translates + cylinder rotates

$$\vec{X} = \vec{X}_{c.center} + \vec{X}_{rotation}$$



2) Identify each's time dependence

After time t, angle rotated is ωt

distance traveled is $r\omega t$

$$\vec{X}_{cm}(t) = (r\omega t, r)$$

Angle relative to center is $\theta(t) = \underbrace{-\frac{\pi}{2}}_{\text{because initially point is down from center}} \underbrace{-\omega t}_{\text{why minus?}}$

Because positive angles are C.C.W

$$\vec{X}_{rot}(t) = (r\cos\theta(t), r\sin\theta(t))$$
$$= \left(r\cos\left(-\frac{\pi}{2} - \omega t\right), r\sin\left(-\frac{\pi}{2} - \omega t\right)\right)$$

$$\boxed{\vec{X} = \left(r\omega t + r\cos\left(-\frac{\pi}{2} - \omega t\right), r + r\sin\left(-\frac{\pi}{2} - \omega t\right)\right) \quad -\infty < t < \infty}$$

Velocity, Speed, Acceleration

If $\vec{X}(t)$ is parameterization of position wrt time,

$\vec{V}(t) = \frac{d}{dt}\vec{X}(t)$ is velocity (vector) at time t

$|\vec{V}(t)|$ is speed (scalar) at time t

$\vec{a}(t) = \frac{d^2\vec{X}(t)}{dt^2} = \frac{d\vec{V}(t)}{dt}$ is acceleration

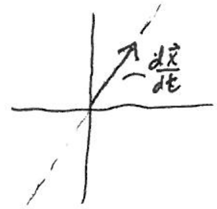
Note: $\vec{V}(t)$ provides tangent vector to curve

Example: Find velocity, speed, acceleration of $\vec{X}(t) = (2t, 3t)$

$\frac{d\vec{X}}{dt} = (2, 3)$ — velocity

$\left|\frac{d\vec{X}}{dt}\right| = \sqrt{2^2+3^2} = \sqrt{13}$ — speed

$\frac{d^2\vec{X}}{dt^2} = (0, 0)$ — acceleration vector



Example: Same for circular motion (radius r, w rad/sec)

$\vec{X}(t) = (r\cos wt, r\sin wt)$

$\frac{d\vec{X}}{dt}(t) = (-rw\sin wt, rw\cos wt)$

$\left|\frac{d\vec{X}}{dt}\right|(t) = \sqrt{r^2w^2(\sin^2 wt + \cos^2 wt)} = rw$ — units work out

$\frac{d^2\vec{X}}{dt^2}(t) = (-r^2w^2\cos wt, -r^2w^2\sin wt)$

$= -r^2w^2(\cos wt, \sin wt)$ — allways points toward origin