# Lecture 14    5 august 2013

## Surface Integrals

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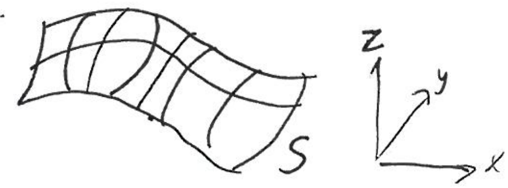
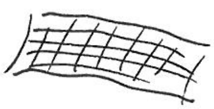



$$\iint_S f(x,y,z)\,dS$$

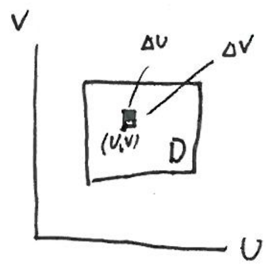
area weighted sum of f along S.

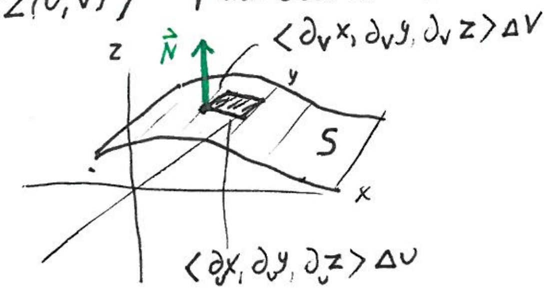
Break surface up into little ~~squares~~ parallelograms.



Add up $f(x,y,z)\cdot$ area of ~~squares~~ parallelograms

Let $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ parameterize surface



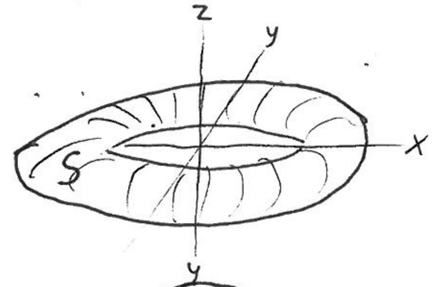

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

Area of parallelogram in $x,y,z$

$$A \approx \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \Delta v$$

So
$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\,dv$$

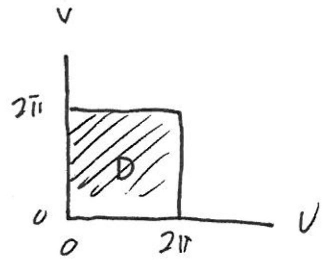
# Example: Surface Area of Torus radii $R$ & $r$

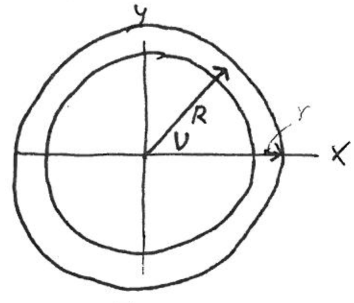


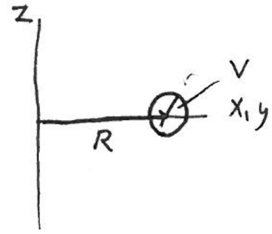

Parametrize by two angles $U, V$

$U$ – angle in $xy$ plane

$V$ – angle in ~~plane~~ cross section




$$x(U,V) = (R + r\cos V)\cos U$$
$$y(U,V) = (R + r\cos V)\sin U$$
$$z(U,V) = r\sin V$$

$$A = \iint_S dS = \iint_D \underbrace{|\partial_u\vec{r} \times \partial_v\vec{r}|}_{N}\, du\, dv$$

$$\vec{N} = \left[-(R + r\cos V)\sin U\,\hat{\imath} + (R + r\cos V)\cos U\,\hat{\jmath}\right]$$
$$\times \left[-r\sin V\cos U\,\hat{\imath} - r\sin V\sin U\,\hat{\jmath} + r\cos V\,\hat{k}\right]$$

$$= \langle r(R + r\cos V)\cos U\cos V,\ r(R + r\cos V)\cos V\sin U,\ r(R + r\cos V)\sin V\rangle$$

$$= r(R + r\cos V)\langle \cos U\cos V,\ \sin U\cos V,\ \sin V\rangle$$

$$|\vec{N}| = r(R + r\cos V)$$

$$A = \int_0^{2\pi}\int_0^{2\pi} r(R + r\cos V)\, du\, dv = 2\pi r\int_0^{2\pi}(R + r\cos V)\, dv$$
$$= 2\pi r\left[2\pi R + r\sin V\Big|_0^{2\pi}\right]$$
$$= 4\pi^2 rR$$
$$= \boxed{2\pi r \cdot 2\pi R}$$

Makes sense