Lecture 13 2 august 2013

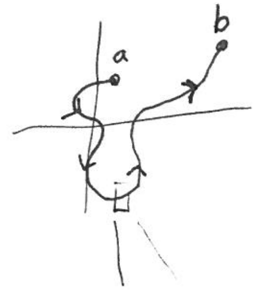
Fundamental Theorem of Line Integrals

Green's Theorem

Fundamental theorem of Line Integrals

$$\int_C \nabla\varphi \cdot d\vec{r} = \phi(b) - \phi(a)$$

If C is any path connecting a & b

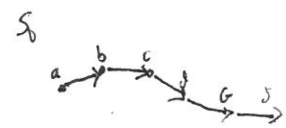


Justification:

$$\int_C \underbrace{\nabla\varphi \cdot \vec{T}}_{} \, ds$$

directional derivative of φ in direction of curve.

$$\nabla\varphi \cdot \vec{T}\, ds \simeq \Delta\varphi$$

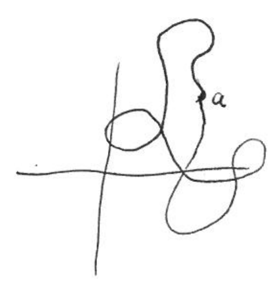


$$\varphi(b) - \varphi(a) + \varphi(c) - \varphi(b) + \dots + \varphi(f) - \varphi(e) = \varphi(f) - \varphi(a)$$

Compare to Fundamental Thm of Calc $\int_a^b f'(t)\,dt = f(b) - f(a)$

Implication: ~~Ar~~

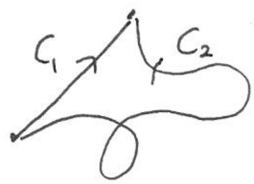
Line Integral of conservative vector fields over closed curves is 0.



$$\oint_C \nabla\varphi \cdot d\vec{r} = \varphi(a) - \varphi(a) = 0$$

line integral over closed curve with positive orientation

Line integrals of conservative vector fields are path independent:



Example: Find $\int_C \vec{F} \cdot d\vec{r}$

for $\vec{F} = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$ for C is circular arc from $(R,0)$ to $(0,R)$.

Method I: Fundamental Theorem

$\vec{F} = \nabla \sqrt{x^2+y^2}$ So $\varphi(x,y) = \sqrt{x^2+y^2}$.

$$\int_C \vec{F} \cdot dr = \varphi(0,R) - \varphi(R,0) = R - R = 0.$$

Method II: Directly

$0 \leq \theta \leq \pi/2$

$\vec{r}(\theta) = \langle R\cos\theta, R\sin\theta \rangle$ $\quad \dfrac{d\vec{r}}{d\theta} = \langle -R\sin\theta, R\cos\theta \rangle$

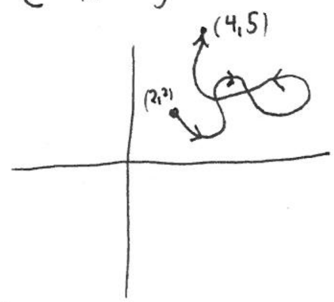
$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \frac{\langle R\cos\theta, R\sin\theta \rangle}{R} \cdot \langle -R\sin\theta, R\cos\theta \rangle \, d\theta$$

$$= R \int_0^{\pi/2} \underbrace{\langle \cos\theta, \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle}_{0} \, d\theta$$

$$= 0.$$

Example: Find $\int_C \vec{F} \cdot d\vec{s}$ for $\vec{F} = x\hat{\imath} + y\hat{\jmath}$

& C is any curve from (2,2) to (4,5)



Notice $\vec{F} = \nabla\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right)$

$\varphi = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$$\int_C \vec{F} \cdot d\vec{s} = \phi(4,5) - \phi(2,2)$$

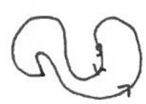
$$= \frac{1}{2}(4)^2 + \frac{1}{2}(5)^2 - \left(\frac{1}{2}2^2 + \frac{1}{2}2^2\right)$$

$$= \boxed{\frac{33}{2}}$$

Closed, Simple, Positively Oriented Curves

A curve C is parameterized is $\vec{r}(t)$ for $a \leq t \leq b$

- C is closed if $\vec{r}(b) = \vec{r}(a)$ "starts where it ends"
- C is simple if it has no self intersections



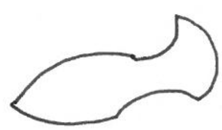
Simple

Not simple

- positively oriented if interior of shape is on left. (counterclockwise)

Positively oriented

- piecewise smooth:



Yes

No

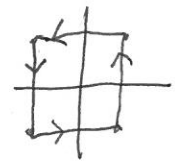
Green's Theorem

Let C be positively oriented, piecewise smooth simple closed curve that bounds region R

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\partial_x Q - \partial_y P\right) dA$$

- Converts between line integrals & area integrals

Example: $\oint_C (x+y^2)\,dx + (y+x^2)\,dy$ w/ $C =$ square with vertices $(\pm 1, \pm 1)$



$P = x + y^2$

$Q = y + x^2$

By Greens Thm

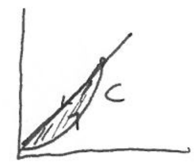
$$\oint_C P\,dx + Q\,dy = \iint_R (\partial_x Q - \partial_y P)\,dA$$

$$= \iint_R (2x - 2y)\,dA$$

$$= 2\iint_R x\,dA - 2\iint_R y\,dA$$

$$= 0 - 0 = 0$$

Example



$$\oint_C (x^2-y^2)\,dx + (xy)\,dy$$

$$\oint_c P\,dx+Q\,dy = \iint_R \left(\partial_x Q - \partial_y P\right) dA$$

$$= \iint_R (y - 2y)\,dA$$

$$= -\iint_R y\,dA = -\int_0^1 \int_{x^2}^{x} y\,dy\,dx$$

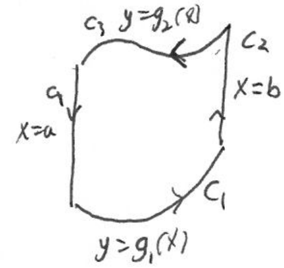
$$= -\int_0^1 \frac{1}{2}x^2 - \frac{x^4}{2}\,dx$$

$$= -\frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right|_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = -\frac{1}{15}$$

Proof of Green's Theorem in Simple Regions

Suppose R is vertically simple



$$\oint_C P\,dx = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} P\,dx$$

On C_2 & C_4 $dx=0$, so $\int_{C_2} P\,dx = 0$

$\int_{C_4} P\,dx = 0$

$$\oint_C P\,dx = \int_{C_1} P\,dx + \int_{C_3} P\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

$$= -\int_a^b P(x, g_2(x)) - P(x, g_1(x)) \quad dx$$

$$= -\int_a^b \int_{g_1(x)}^{g_2(x)} \partial_y P(x,y)\,dy\,dx = -\iint_R \partial_y P\,dA$$

Similarly $\int_C Q\,dy = \iint_R \partial_x Q\,dA$ for horizontally simple regions

Proof in complicated domains: Break into simple subregions