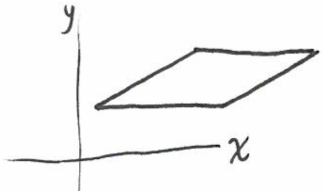
Lecture 11 29 july 2013

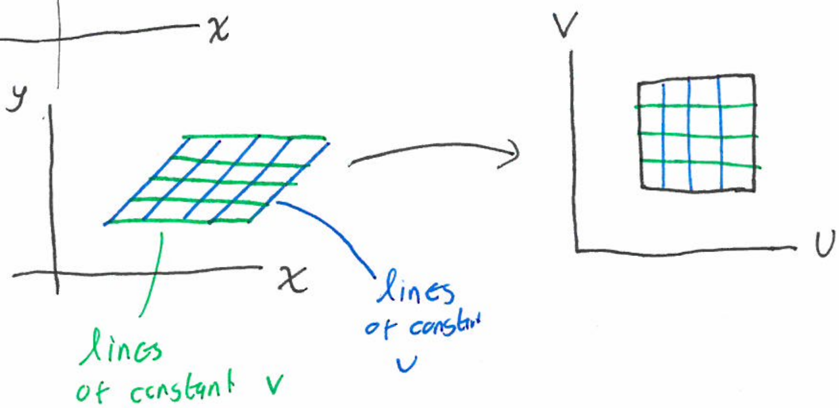
General Change of Coordinates

General Change of Coordinates

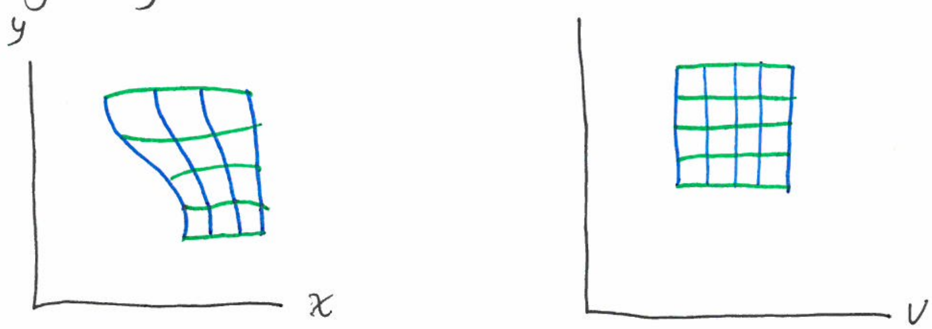
Some domains suggest alternate coordinates



might be easier to integrate with coordinates in directions $\nearrow$ & $\longrightarrow$

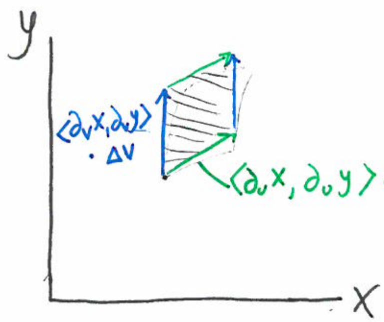


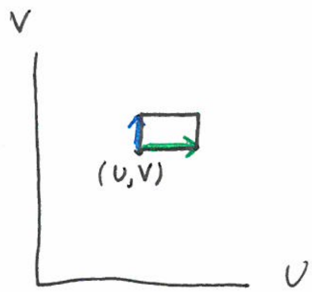
More generally



Area Element in General Coordinates

Let $(x, y) = (f(u,v), g(u,v))$ specify a change of variables





If we change u by Δu & keep v fixed, (x,y) changes by $\left(\partial_u f(u,v), \partial_u g(u,v)\right)\Delta u$

If change v by Δv & keep u fixed, (x,y) changes by $\left(\partial_v f(u,v), \partial_v g(u,v)\right)\Delta v$

Region in xy plane has area $\approx$ area of parallelogram

$$\Delta A = \left| \langle \partial_u x, \partial_u y, 0\rangle \times \langle \partial_v x, \partial_v y, 0\rangle \right|$$

$$= \underset{\substack{\uparrow \\ \text{length} \\ \text{of vector}}}{\Bigg|} \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_u x & \partial_u y & 0 \\ \partial_v x & \partial_v y & 0 \end{vmatrix}}_{\text{det of matrix is vector}} \Bigg| = \left| \vec{k}\,(\partial_u x\, \partial_v y - \partial_u y\, \partial_v x) \right|$$

$$= \underset{\substack{\uparrow \\ \text{absolute} \\ \text{value}}}{\Bigg|} \underbrace{\begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix}}_{\text{det of matrix}} \Bigg|$$

Let $J = \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix} = \dfrac{\partial(x,y)}{\partial(u,v)}$ is "Jacobian"

determinant

$$dA = \underset{\substack{\uparrow \\ \text{absolute} \\ \text{value of } J}}{|J|}\, du\, dv$$

Examples:

Compute the Jacobian for polar coords

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$J = \begin{vmatrix} \partial_r x & \partial_r y \\ \partial_\theta x & \partial_\theta y \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

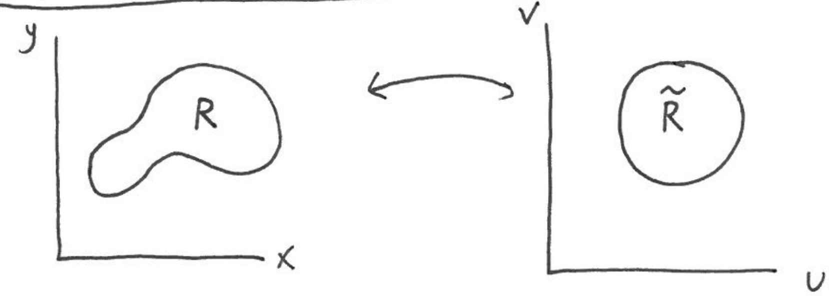
Compute the Jacobian for spherical coords

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$

$$J = \begin{vmatrix} \partial_r x & \partial_r y & \partial_r z \\ \partial_\varphi x & \partial_\varphi y & \partial_\varphi z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= r^2\sin\varphi$$

Change of Variables Theorem



Let $x = f(u,v)$
$y = g(u,v)$.

Let $\tilde{R}$ be corresponding set of R in (u,v) plane

$$\iint_R F(x,y)\,dx\,dy = \iint_{\tilde{R}} \underbrace{F(f(u,v), g(u,v))}_{\text{Express integrand in terms of } u \,\&\, v} \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian (absolute value)}} du\,dv$$

Comments:

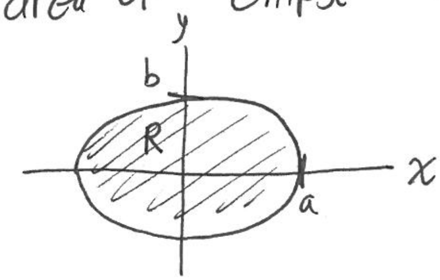
- $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix} = \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} = \left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$

- Remember it is $\dfrac{\partial(x,y)}{\partial(u,v)}$ not $\dfrac{\partial(u,v)}{\partial(x,y)}$ because

$$dx\,dy \approx \left|\frac{\partial(x,y)}{\cancel{\partial(u,v)}}\right| \cancel{du}\,dv$$

- Absolute value of J!

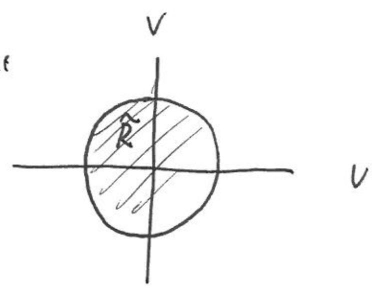
Example: Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$A = \iint_R 1 \, dx\,dy$$

Change coordinates $u = \frac{x}{a}$, $v = \frac{y}{b}$ so $x = au$, $y = bv$

In uv plane



$$A = \iint_R 1 \, dx\,dy = \iint_{\hat{R}} 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

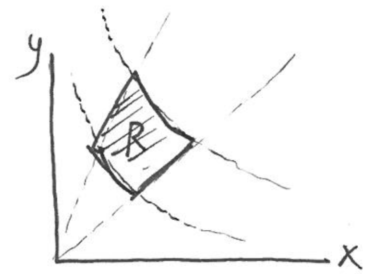
$$A = \iint_{\hat{R}} ab \, du\,dv = ab \cdot \iint_{\hat{R}} du\,dv$$

$= ab \cdot$ area of unit circle

$= \pi ab$

Example: Find area of region between

$$y = x \qquad xy = 1$$
$$y = 2x \qquad xy = 2$$

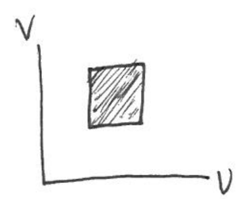


$$A = \iint_R 1 \, dx\,dy$$

Choose coords: $u = y/x$, $v = xy$

$1 \leq u \leq 2$

$1 \leq v \leq 2$



$$A = \iint_{\tilde{R}} 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Compute: $\dfrac{\partial(x,y)}{\partial(u,v)} = \left(\dfrac{\partial(u,v)}{\partial(x,y)} \right)^{-1}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{vmatrix} = \begin{vmatrix} -y/x^2 & y \\ 1/x & x \end{vmatrix} = -2\frac{y}{x} = -2u$$

So $A = \displaystyle\int_1^2\int_1^2 \frac{1}{2u}\, du\,dv = \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{\log 2}{2}.$