Generative Adversarial Networks

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Outline

- GANs - examples and properties
- Minimax formulation and theory
- Wasserstein GANs
- Challenges

Generative Adversarial Networks

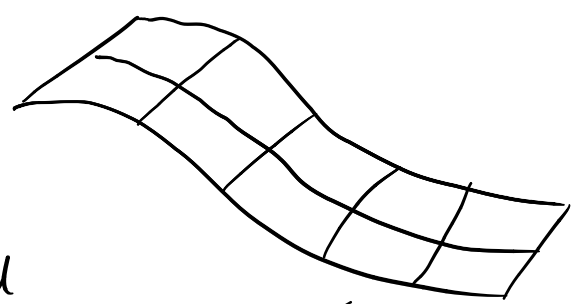
(Goodfellow et al. 2014)

Generative model trained in a game-theoretic adversarial way

$G: \mathbb{R}^k \to \mathbb{R}^n$ st if $z \sim N(0, I_k)$ then $G(z)$ samples from a learned data distribution

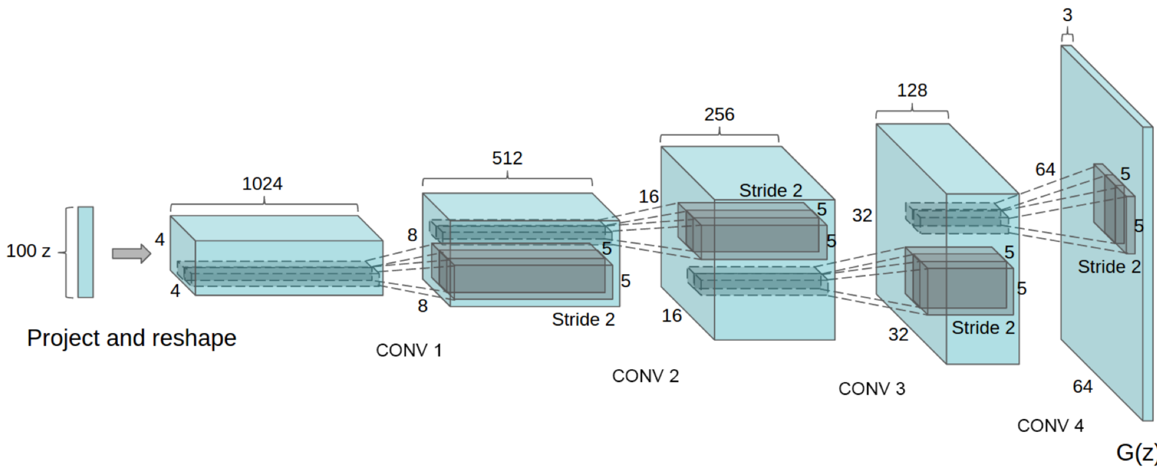
latent space image space

While G induces a distribution on $\mathbb{R}^n$, we will not attempt to maximize data likelihood



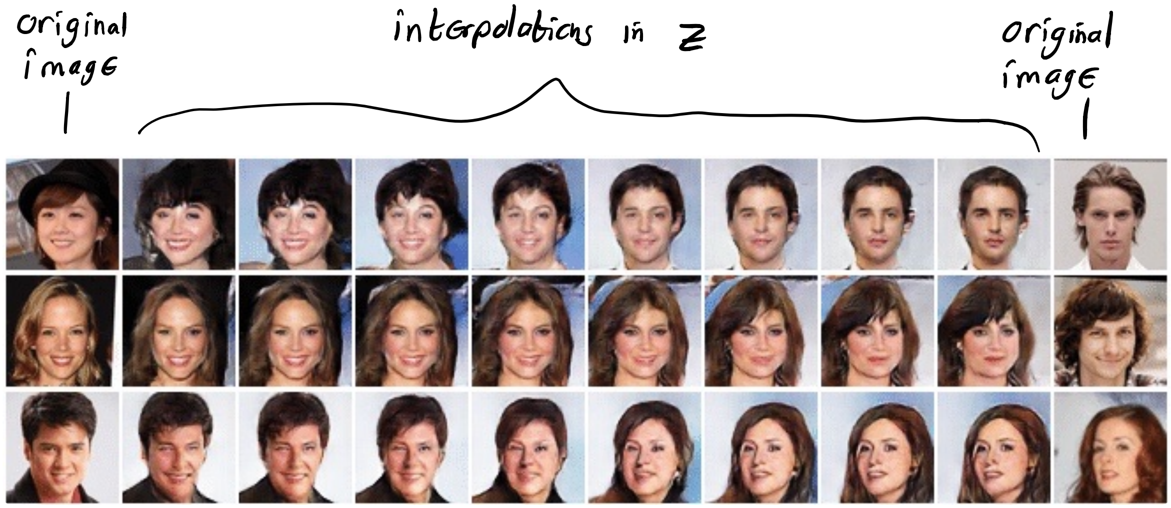
Range(G)

Example architecture (DCGAN) (Radford et al. 2016)



Synthetic Samples when trained on LSUN Bedrooms:

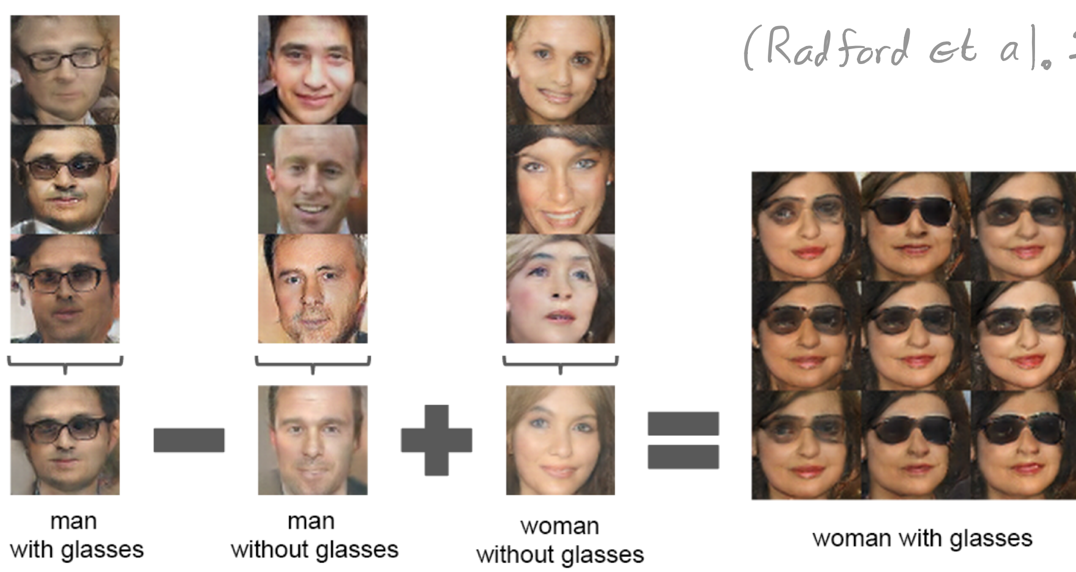
Can interpolate in latent space:



(Ulyanov et al. 2017)

There is semantically meaningful arithmetic in latent space:

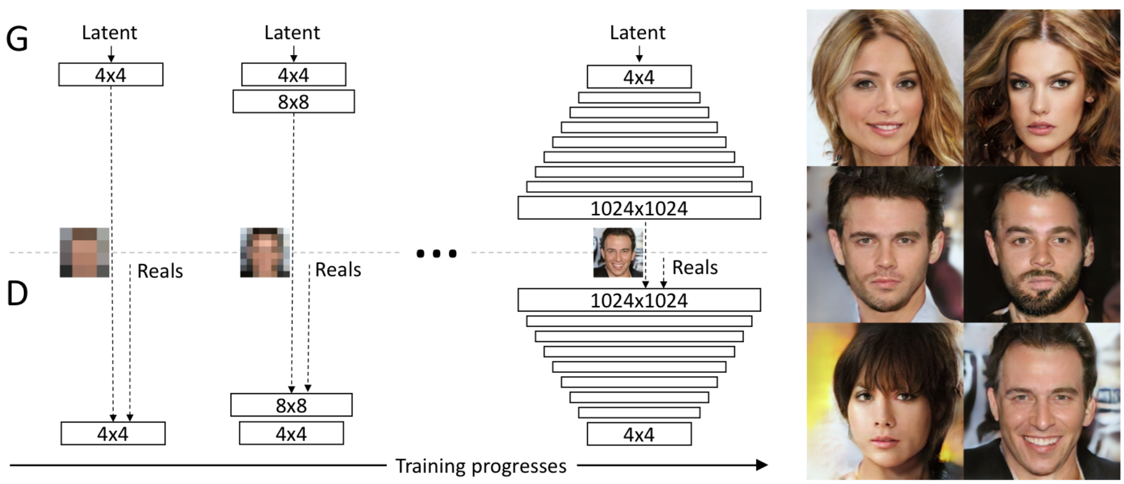
(Radford et al. 2016)



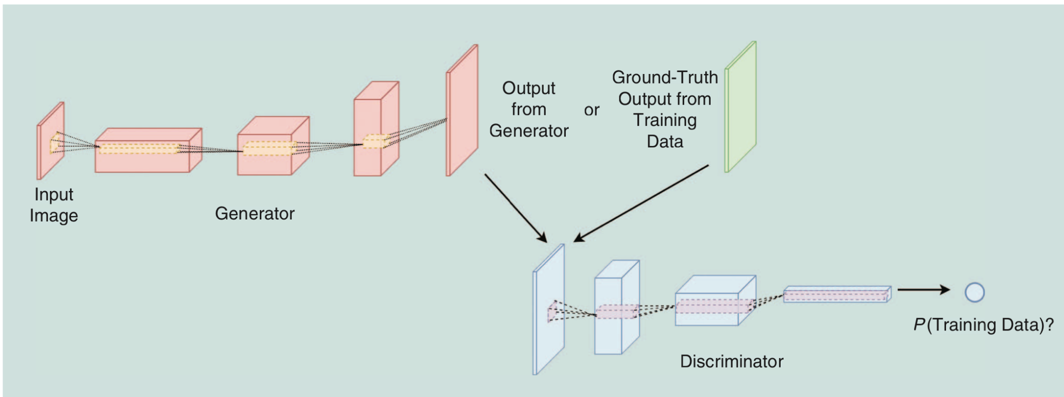
There is a direction in z corresponding to having glasses

GANs have been trained that can generate photorealistic faces

(Korras et al. 2018)



Idea: Train a model by trying to fool a concurrently trained discriminator



(Lucas et al. 2018)

Formulation of GAN training as minimax optimization

Let P_d denote data distribution
P_z be $\mathcal{N}(0, I_k)$

Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be the generator
$D: \mathbb{R}^n \to [0,1]$ be $P(\text{input is real})$

Value function

$$V(D,G) = \mathbb{E}_{x \sim P_d} \log D(x) + \mathbb{E}_{z \sim P_z} \log\left(1 - D(G(z))\right)$$

Why optimize this?

it is the negative cross-entropy loss
but label = real when $x \sim p_d$
and label = not real when $z \sim p_z$

Cross entropy loss

$$\ell_{CE}(p, q) = -\sum_{s \in S} p(s) \log q(s) = -\mathbb{E}_p(\log q)$$

r.v.s over S

Minimax formulation

$$\min_G \max_D \; \mathbb{E}_{x \sim p_d} \log D(x) + \mathbb{E}_{z \sim p_z} \log(1 - D(G(z)))$$

D wants to maximize neg. cross-entropy

G wants the opposite

Minibatch Stochastic Gradient Descent Algorithm

Algorithm 1 Minibatch stochastic gradient descent training of generative adversarial nets. The number of steps to apply to the discriminator, k, is a hyperparameter. We used $k = 1$, the least expensive option, in our experiments.

for number of training iterations **do**
  for k steps **do**
  - Sample minibatch of m noise samples $\{\boldsymbol{z}^{(1)}, \dots, \boldsymbol{z}^{(m)}\}$ from noise prior $p_g(\boldsymbol{z})$.
  - Sample minibatch of m examples $\{\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(m)}\}$ from data generating distribution $p_{\text{data}}(\boldsymbol{x})$.
  - Update the discriminator by ascending its stochastic gradient:

$$\nabla_{\theta_d} \frac{1}{m} \sum_{i=1}^{m} \left[\log D\left(\boldsymbol{x}^{(i)}\right) + \log\left(1 - D\left(G\left(\boldsymbol{z}^{(i)}\right)\right)\right) \right].$$

  end for
  - Sample minibatch of m noise samples $\{\boldsymbol{z}^{(1)}, \dots, \boldsymbol{z}^{(m)}\}$ from noise prior $p_g(\boldsymbol{z})$.
  - Update the generator by descending its stochastic gradient:

$$\nabla_{\theta_g} \frac{1}{m} \sum_{i=1}^{m} \log\left(1 - D\left(G\left(\boldsymbol{z}^{(i)}\right)\right)\right).$$

end for
The gradient-based updates can use any standard gradient-based learning rule. We used momentum in our experiments.

— perhaps multiple updates to D per update to G

(Goodfellow et al. 2014)

Why is the GAN value function the right thing to optimize?

Claim: For fixed G, the optimal D is

$$D_G^*(x) = \frac{P_d(x)}{P_d(x) + P_g(x)}$$

Proof:

$$V(G,D) = \mathop{\mathbb{E}}_{x\sim P_d} \log D(x) + \mathop{\mathbb{E}}_{z\sim P_z} \log(1 - D(G(z)))$$

$$= \mathop{\mathbb{E}}_{x\sim P_d} \log D(x) + \mathop{\mathbb{E}}_{x\sim P_g} \log(1 - D(x))$$

distribution induced by generator

$$= \int_{\mathbb{X}} \left(P_d(x) \log D(x) + P_g(x) \log (1 - D(x)) \right) dx$$

To find max over D:

Use Variational Calculus and differentiate with respect to D and set equal to 0

$$\frac{P_d(x)}{D(x)} - \frac{P_g(x)}{1 - D(x)} \equiv 0$$

$$\Rightarrow D^*(x) = \frac{P_d(x)}{P_d(x) + P_g(x)} \quad \blacksquare$$

Theorem: The global minimum of

$$C(G) = \max_D V(G, D)$$

is unique and achieved iff $P_g = P_d$.

Proof: By previous claim,

$$C(G) = \mathbb{E}_{x \sim P_d} \log D_G^*(x) + \mathbb{E}_{x \sim P_g} \log (1 - D_G^*(x))$$

$$= \mathbb{E}_{x\sim p_d} \log p_d \frac{2}{p_d+p_g} + \mathbb{E}_{x\sim p_g} \log p_g \frac{2}{p_d+p_g} - \log 4$$

$$= -\log 4 + D_{KL}\left(p_d \,\Big\|\, \frac{p_d+p_g}{2}\right) + D_{KL}\left(p_g \,\Big\|\, \frac{p_d+p_g}{2}\right)$$

nonnegative and
0 iff $p_d = p_g$ $\blacksquare$

Limits on this theory:

Nonparametric, infinite capacity models
(all probability distributions)

Does not assure the minimax problem
can be solved to global optimality

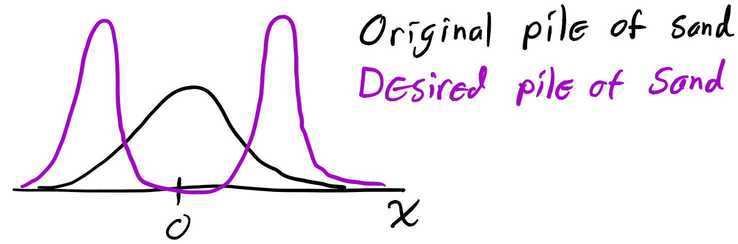
Wasserstein GAN

(Arjovsky et al. 2017)

Goal: minimize "distance" between P_d and P_g

Use Earth mover distance
(Wasserstein-1 distance)

Illustration:



Move each grain such that average distance moved is minimized

Formally,

$$W(P_d, P_g) = \inf_{\gamma \in \Pi(P_d, P_g)} \mathbb{E}_{(x,y)\sim\gamma} \|x-y\|$$

w/ $\Pi(P_d, P_g) = \left\{ \begin{array}{l} \text{joint distributions on } (x,y) \\ \text{s.t. marginals are } P_d \text{ and } P_g \end{array} \right\}$

Why minimize EMD?

Plain GAN (earlier) roughly minimizes

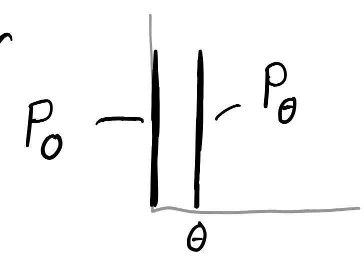
$$D_{KL}\left(P_d \,\|\, \frac{P_d+P_g}{2}\right) + D_{KL}\left(P_g \,\|\, \frac{P_d+P_g}{2}\right) = JS(P_d, P_g)$$

Jensen-Shannon divergence

This is not continuous in P_d and P_g, but EMD is.

Example:

Consider uniform distribution over the 2d line segment
$P_\theta = \{(\theta, y) \mid 0 \le y \le 1\} \subset \mathbb{R}^2$



$$KL(P_0, P_\theta) = \begin{cases} \infty & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

$$JS(P_0, P_\theta) = \begin{cases} \log 2 & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

$$W(P_0, P_\theta) = |\theta|$$

As $\theta \to 0$, only $W(P_0, P_\theta) \to 0$.

Approximating EMD w/ nets

By Kantorovich-Rubinstein duality

$$W(P_d, P_g) = \sup_{\|f\|_L \le 1} \mathbb{E}_{x \sim P_d} f(x) - \mathbb{E}_{x \sim P_g} f(x)$$

Lipschitz constant: $\|f\|_L = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}$

At the expense of a factor of K, can take sup over $\|f\|_L \le K$

To estimate $W(P_d, P_g)$:

$$\max_{w \in \mathcal{W}} \mathbb{E}_{x \sim P_d} f_w(x) - \mathbb{E}_{z \sim P_z} f_w(G_\theta(z))$$

where f_w are neural nets w/ parameters w in a compact set $\mathcal{W}$.

eg each weight is in $[-0.01, 0.01]$

WGAN formulation

$$\min_{W} \max_{\theta} \mathop{\mathbb{E}}_{x \sim p_d} f_w(x) - \mathop{\mathbb{E}}_{z \sim p_z} f_w(G_\theta(z))$$

Call f_w the "critic"

Algorithm:

Algorithm 1 WGAN, our proposed algorithm. All experiments in the paper used the default values $\alpha = 0.00005$, $c = 0.01$, $m = 64$, $n_{\text{critic}} = 5$.

Require: : α, the learning rate. c, the clipping parameter. m, the batch size. n_{critic}, the number of iterations of the critic per generator iteration.

Require: : w_0, initial critic parameters. θ_0, initial generator's parameters.

while θ has not converged **do**

  for $t = 0, ..., n_{\text{critic}}$ **do**

    Sample $\{x^{(i)}\}_{i=1}^m \sim \mathbb{P}_r$ a batch from the real data.

    Sample $\{z^{(i)}\}_{i=1}^m \sim p(z)$ a batch of prior samples.

    $g_w \leftarrow \nabla_w \left[\frac{1}{m}\sum_{i=1}^m f_w(x^{(i)}) - \frac{1}{m}\sum_{i=1}^m f_w(g_\theta(z^{(i)}))\right]$

    $w \leftarrow w + \alpha \cdot \text{RMSProp}(w, g_w)$

    $w \leftarrow \text{clip}(w, -c, c)$

  end for

  Sample $\{z^{(i)}\}_{i=1}^m \sim p(z)$ a batch of prior samples.

  $g_\theta \leftarrow -\nabla_\theta \frac{1}{m}\sum_{i=1}^m f_w(g_\theta(z^{(i)}))$

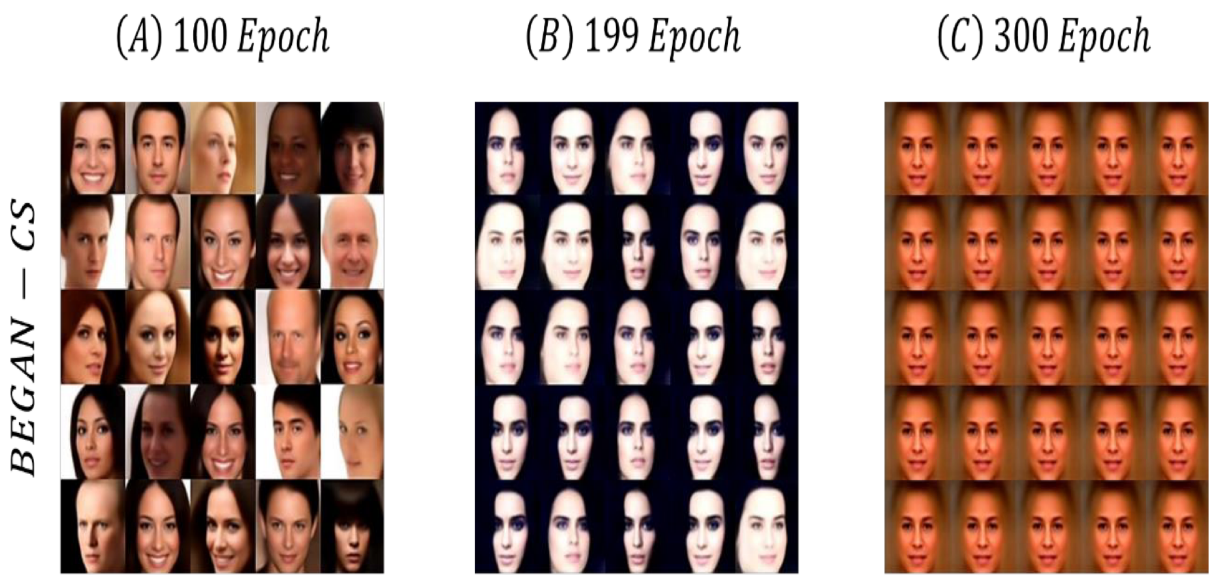
  $\theta \leftarrow \theta - \alpha \cdot \text{RMSProp}(\theta, g_\theta)$

end while

Challenges with GANs:

- Difficulty in training (Eg # D updates per G update)

- Mode collapse



(Park et al. 2020)

- No evaluation metric
- No likelihood estimates
- Difficult to invert

$$\min_{z} \| G(z) - y \|^2$$