CS3000: Algorithms & Data
Paul Hand

Lecture 21:

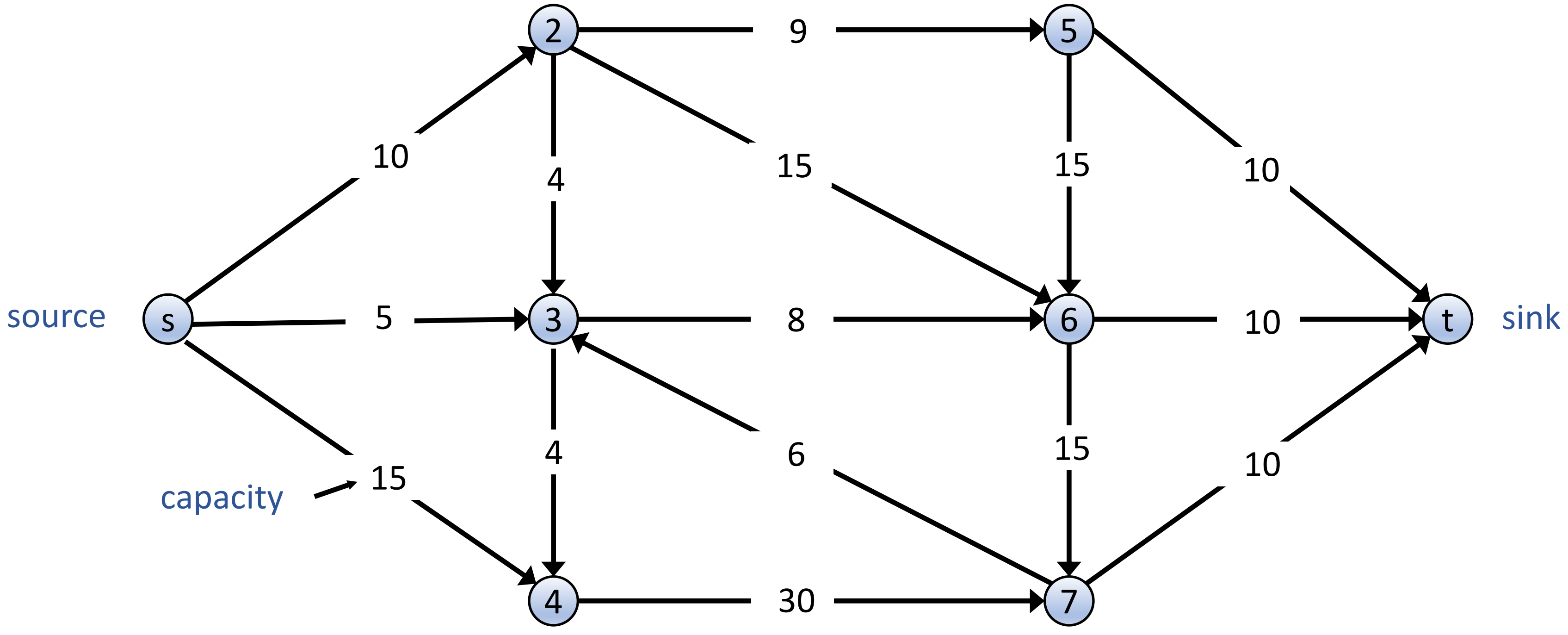
- Network Flow: flows, cuts, duality
- Ford-Fulkerson

Apr 10, 2019

Flow Networks

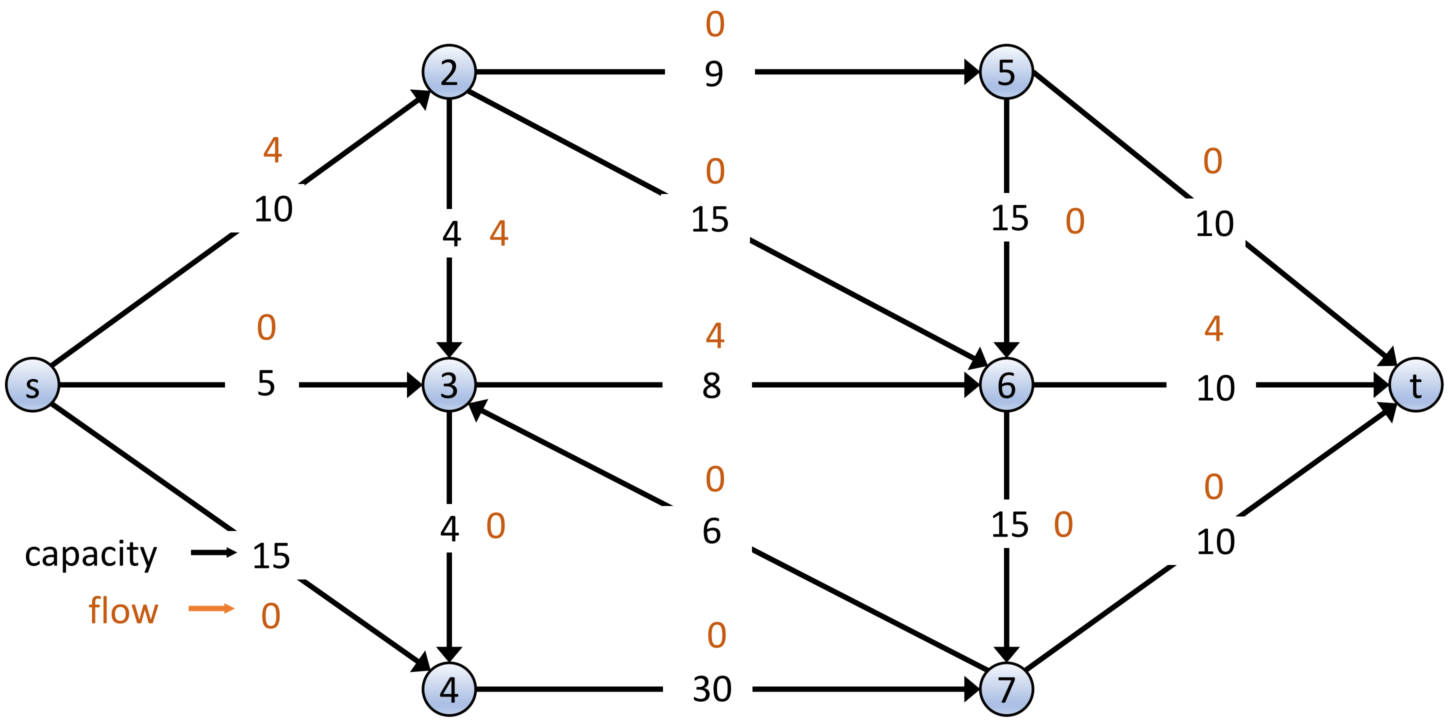
Flow Networks

- Directed graph $G = (V, E)$
- Two special nodes: source s and sink t
- Edge capacities $c(e)$



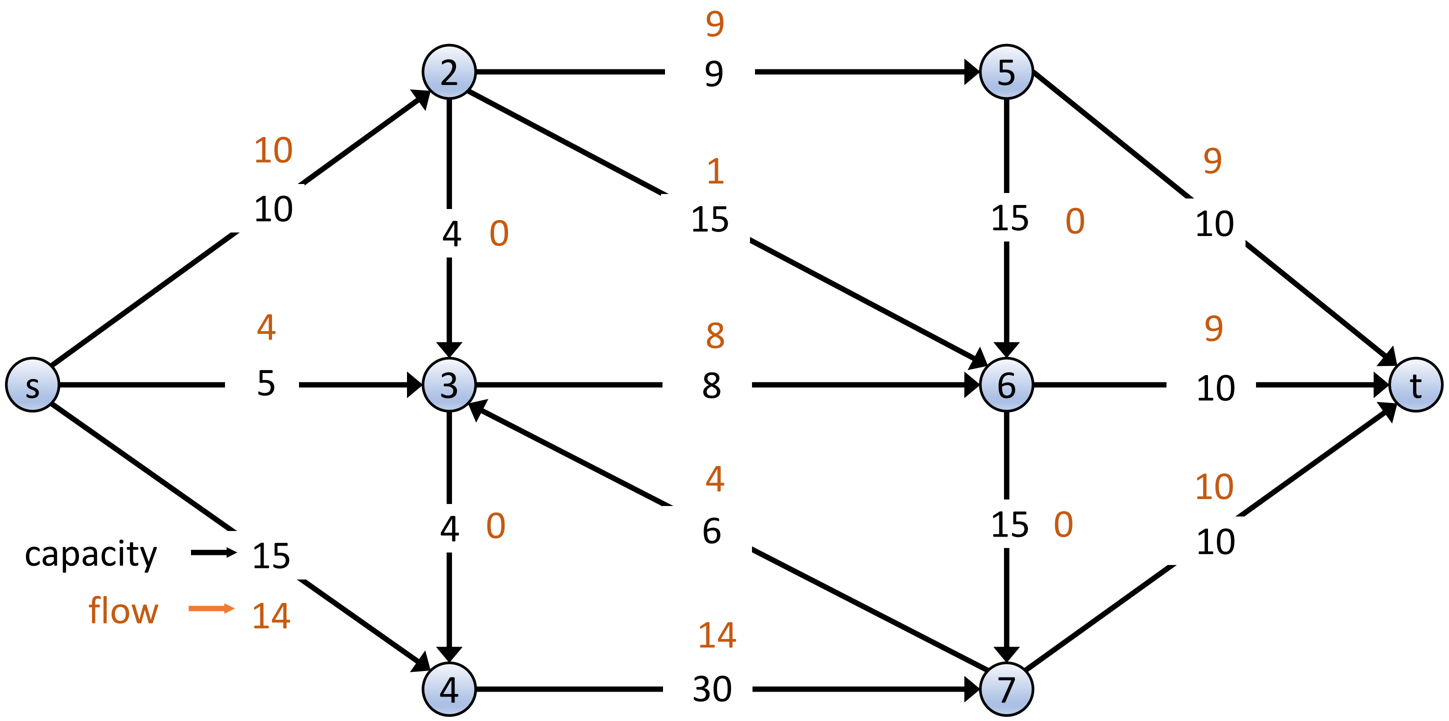
Flows

- An s-t flow is a function $f(e)$ such that
  - For every $e \in E$, $0 \leq f(e) \leq c(e)$ (capacity)
  - For every $v \in V, v \neq s, v \neq t$,
    $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)
- The value of a flow is $val(f) = \sum_{e \text{ out of } s} f(e)$



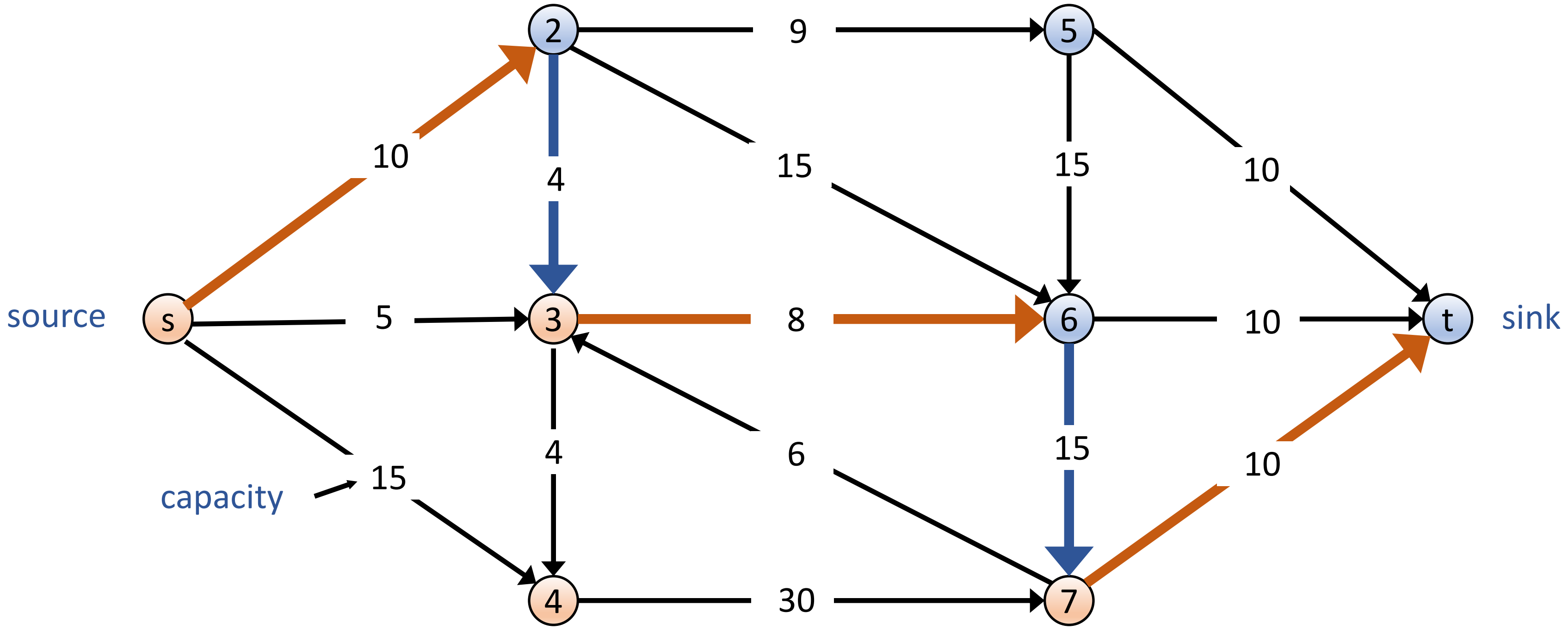
Maximum Flow Problem

- Given G = (V,E,s,t,{c(e)}), find an s-t flow of maximum value



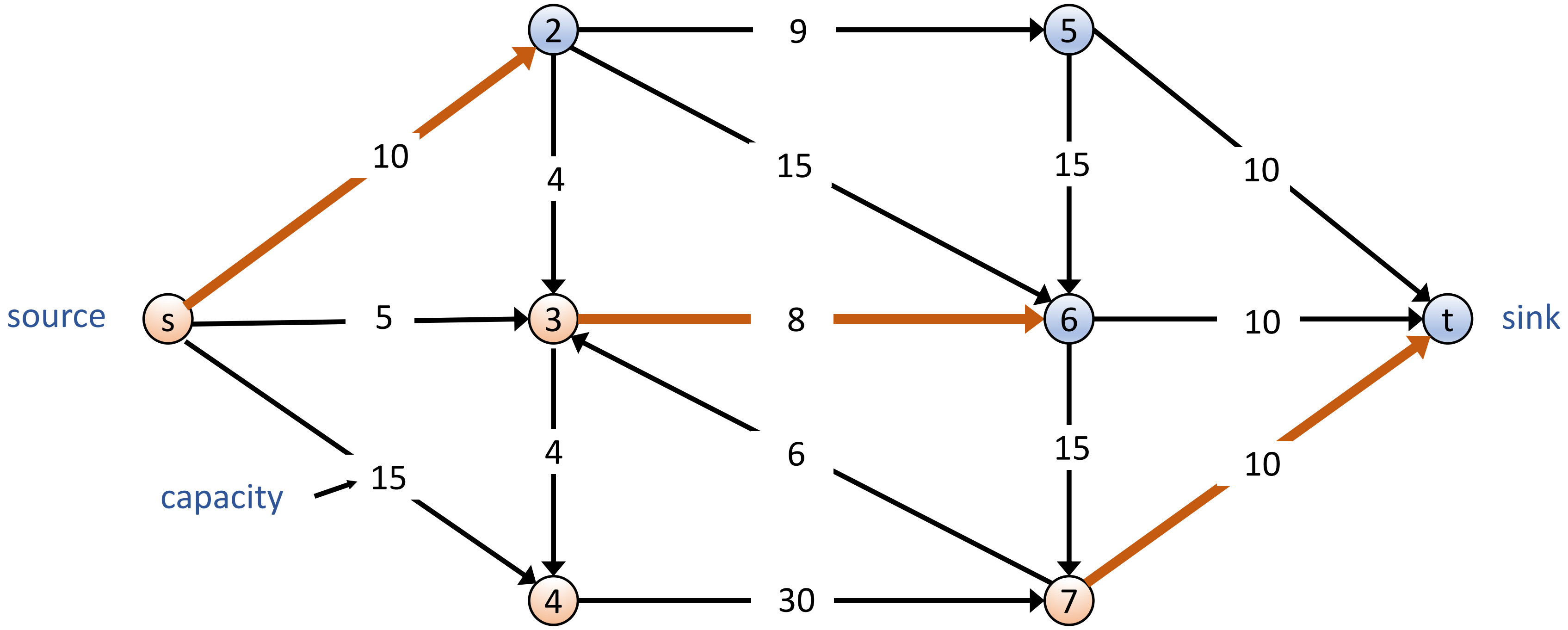
Cuts

- An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$
- The capacity of a cut (A,B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut problem

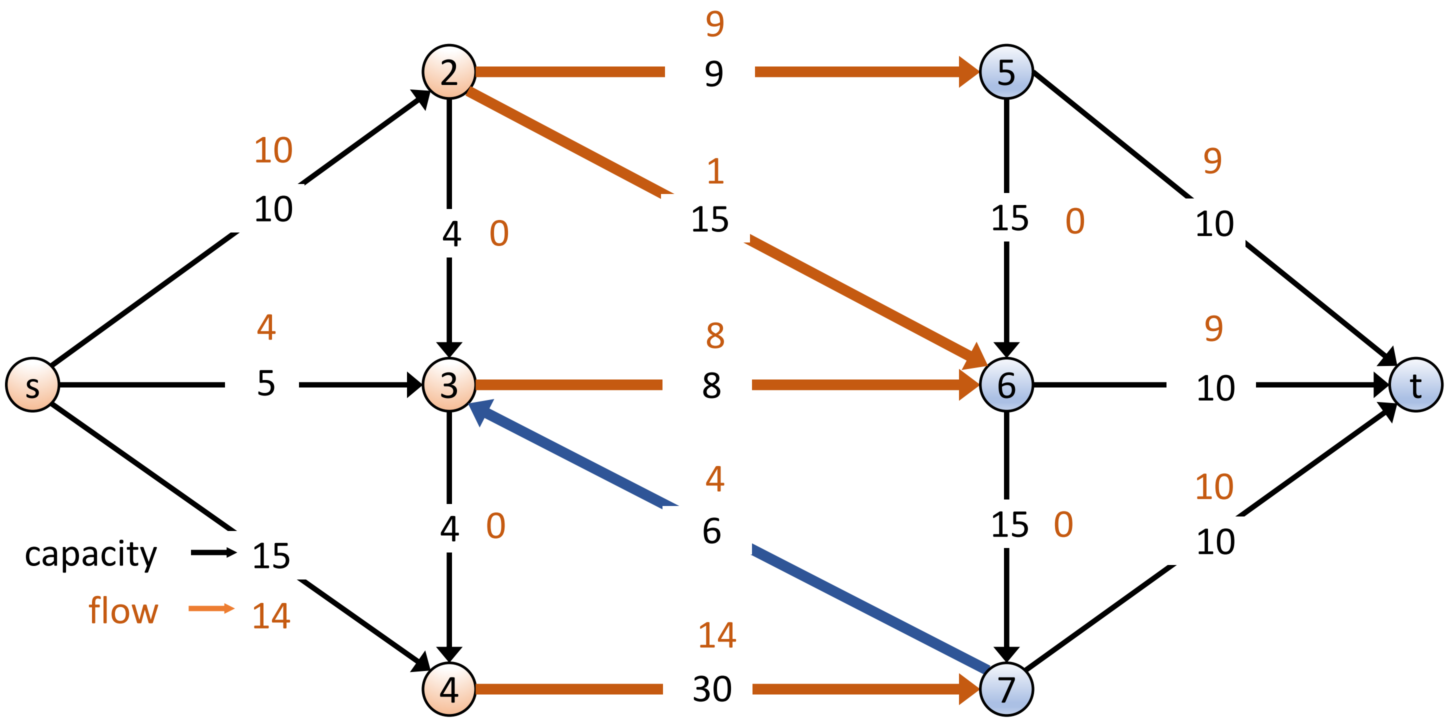
- Given G = (V,E,s,t,{c(e)}), find an s-t cut of minimum capacity



Flows vs. Cuts

- **Fact:** If f is any s-t flow and (A, B) is any s-t cut, then the net flow across (A, B) is equal to the amount leaving s

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = val(f)$$



Max Flow Min Cut Duality

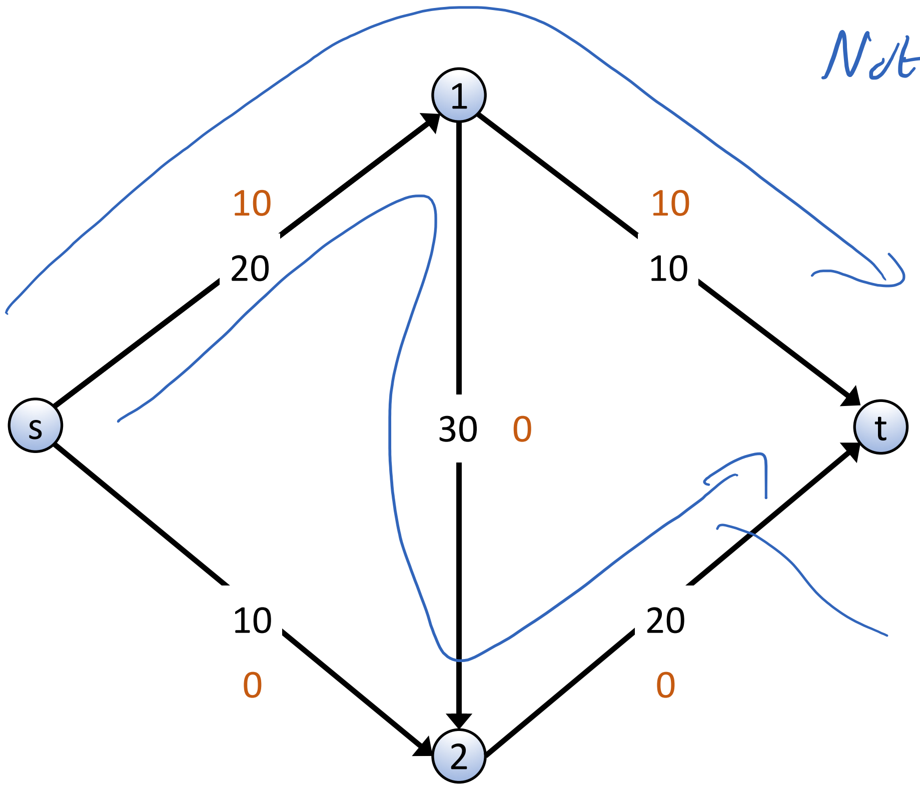
- **Weak Duality:** Let f be any s-t flow and (A, B) any s-t cut,

$$val(f) \leq cap(A, B)$$

- **Proof:**

Augmenting Paths

- Given a network $G = (V, E, s, t, \{c(e)\})$ and a flow f, an **augmenting path** P is an $s \to t$ path such that $f(e) < c(e)$ for every edge $e \in P$



Not augmenting

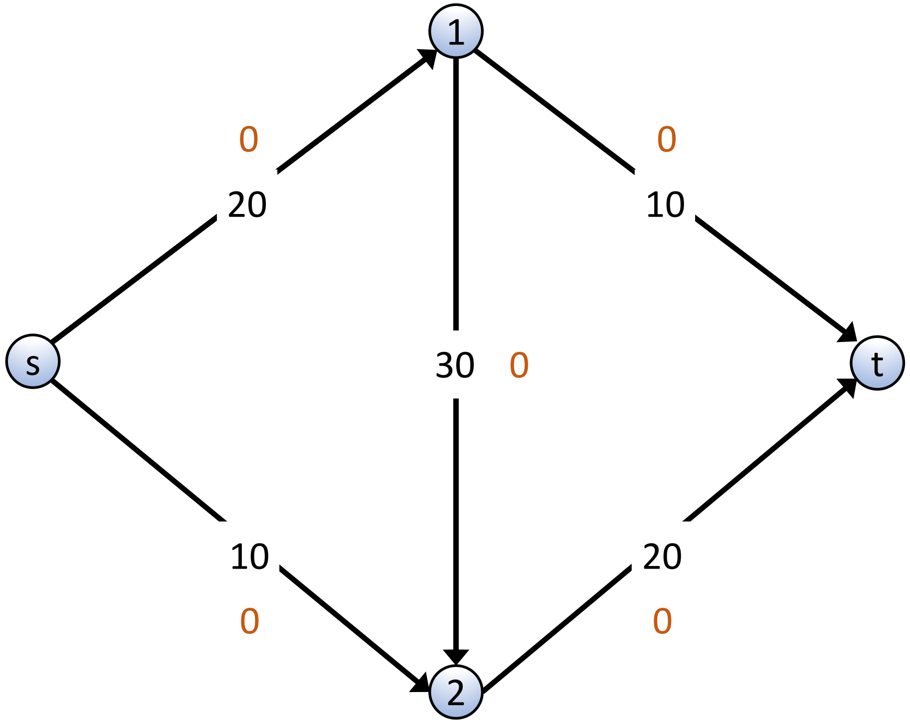
is augmenting

Greedy Max Flow

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P, max it out
- Repeat until you get stuck

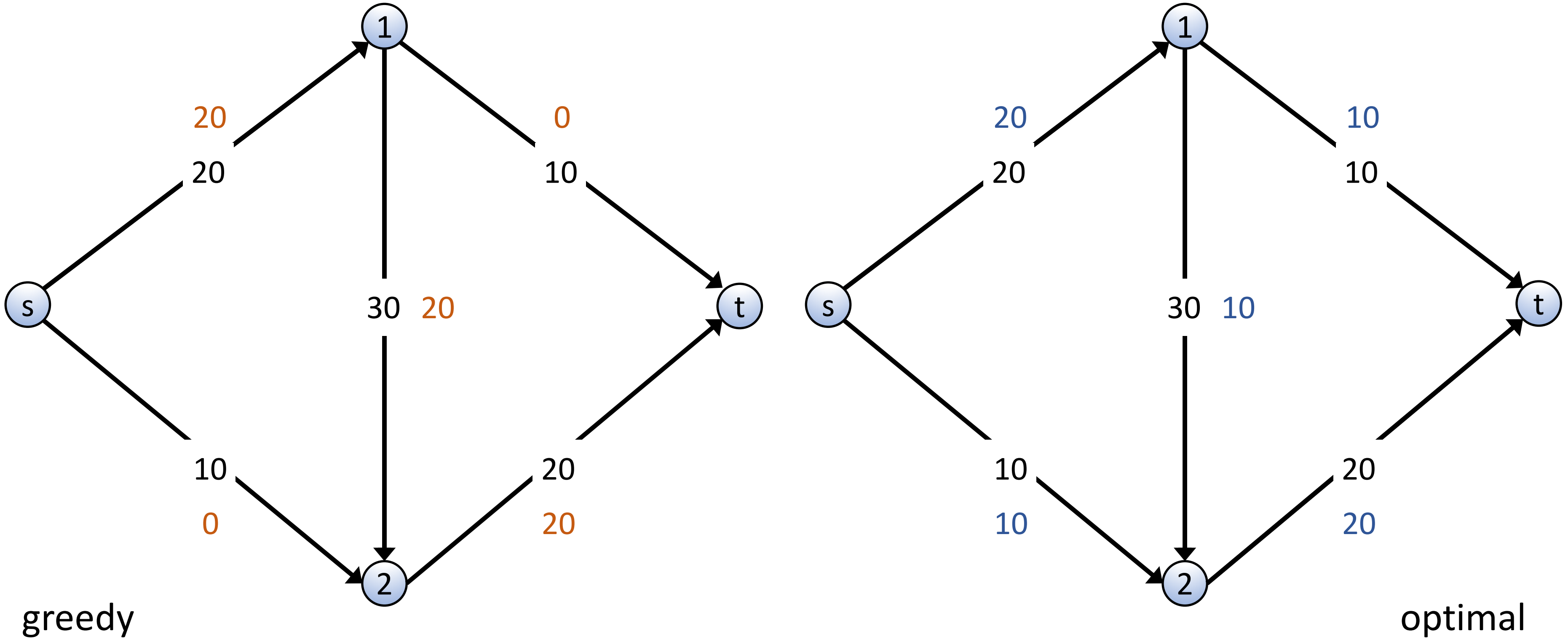
Doesn't work

(Not correct)



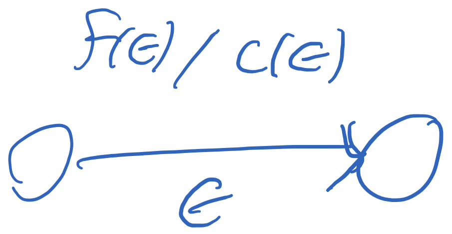
Does Greedy Work?

- Greedy gets stuck before finding a max flow
- How can we get from our solution to the max flow?

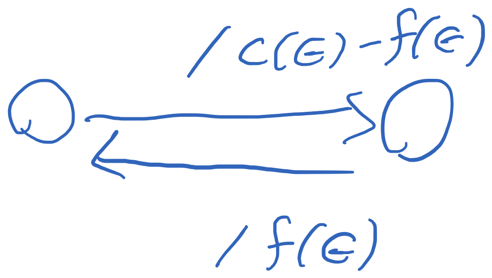


Residual Graphs

- Original edge: $e = (u, v) \in E$.
  - Flow $f(e)$, capacity $c(e)$



- Residual edge
  - Allows “undoing” flow
  - $e = (u, v)$ and $e^R = (v, u)$.
  - Residual capacity



"residual edge"

- Residual graph $G_f = (V, E_f)$
  - Edges with positive residual capacity.
  - $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

all edges below capacity

reverse of any edge w/ flow

Augmenting Paths in Residual Graphs

- Let G_f be a **residual graph**
- Let P be an augmenting path in the **residual graph**
- **Fact:** $f' = \text{Augment}(G_f, P)$ is a valid flow

```
Augment(Gf, P)
    b ← the minimum capacity of an edge in P of Gf
    for e ∈ P
        if e ∈ E:   f(e) ← f(e) + b
        else:       f(e) ← f(e) - b
    return f
```

"Greedy"

Ford-Fulkerson Algorithm

Any path from s→t in G_f is augmenting (otherwise an edge at capacity would have been removed)

```
FordFulkerson(G,s,t,{c})
    for e ∈ E: f(e) ← 0
    G_f is the residual graph

    while (there is an s-t path P in G_f)
        f ← Augment(G_f,P)
        update G_f

    return f
```

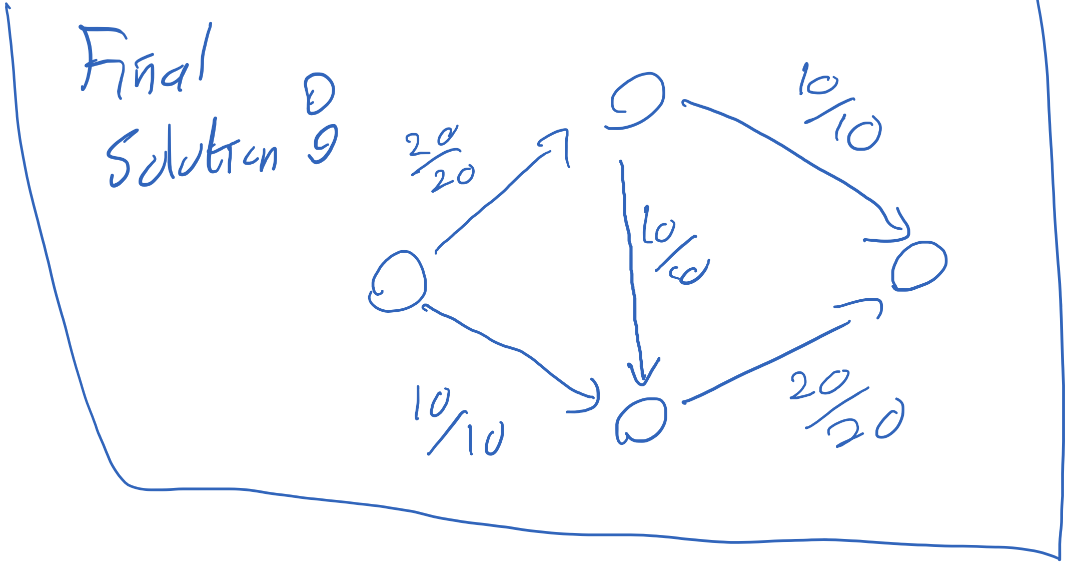
initializes flow at zero

find an arbitrary augmenting path

min is over edges in P

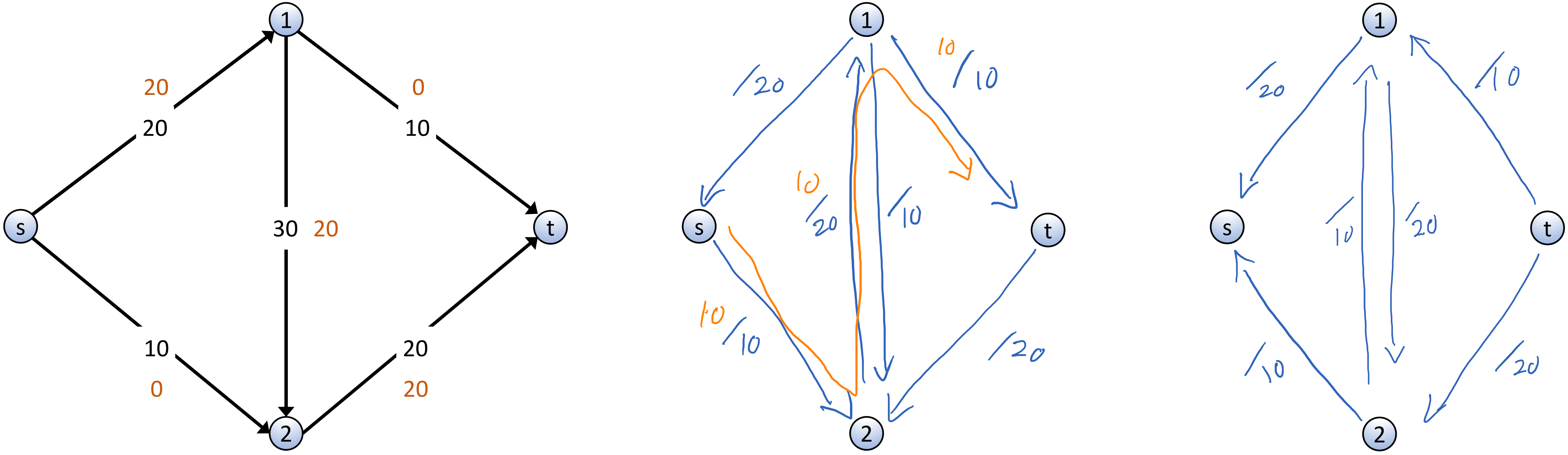
```
Augment(G_f, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if e ∈ E:    f(e) ← f(e) + b
        else:        f(e) ← f(e) - b
    return f
```

Ford-Fulkerson Algorithm



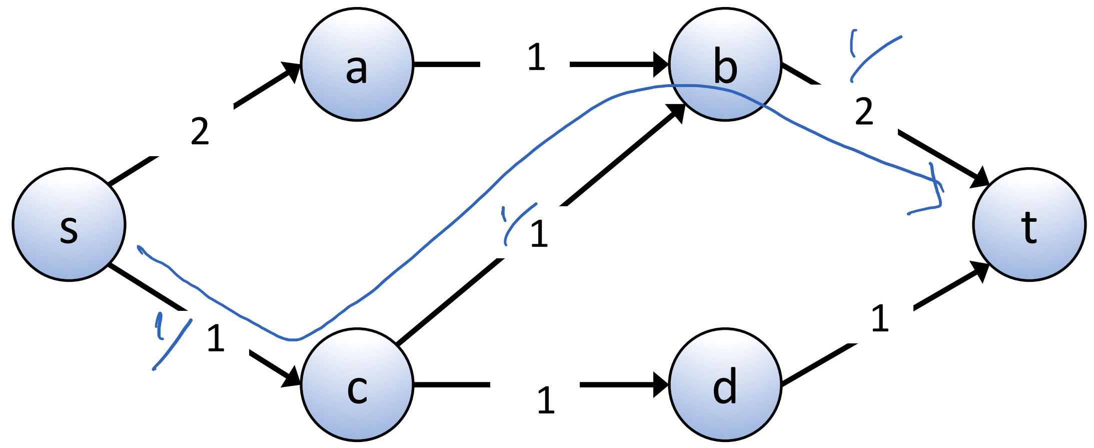
- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Max it out
- Repeat until you get stuck

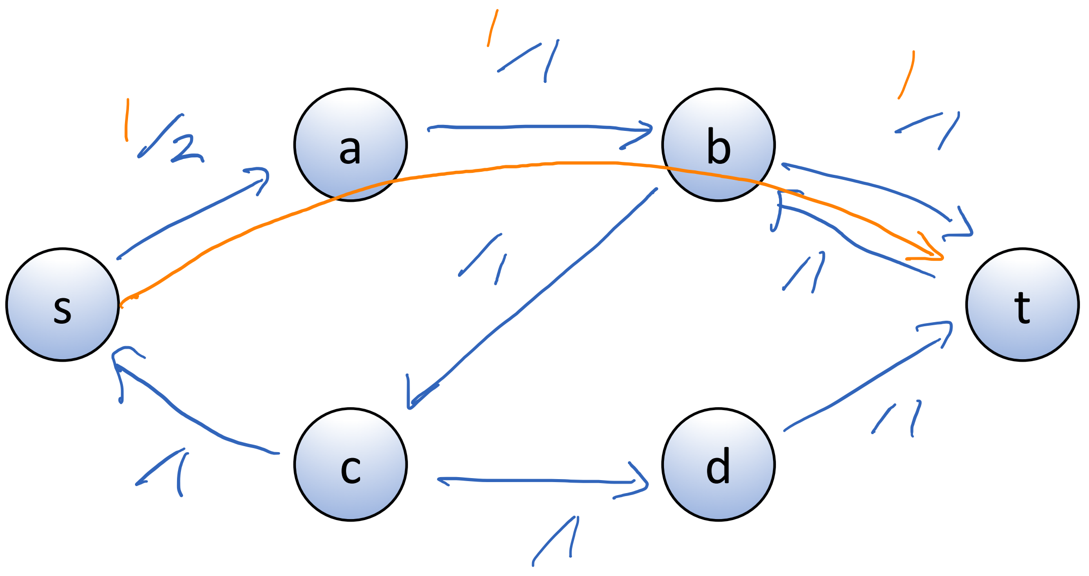
Compute Residual Graph

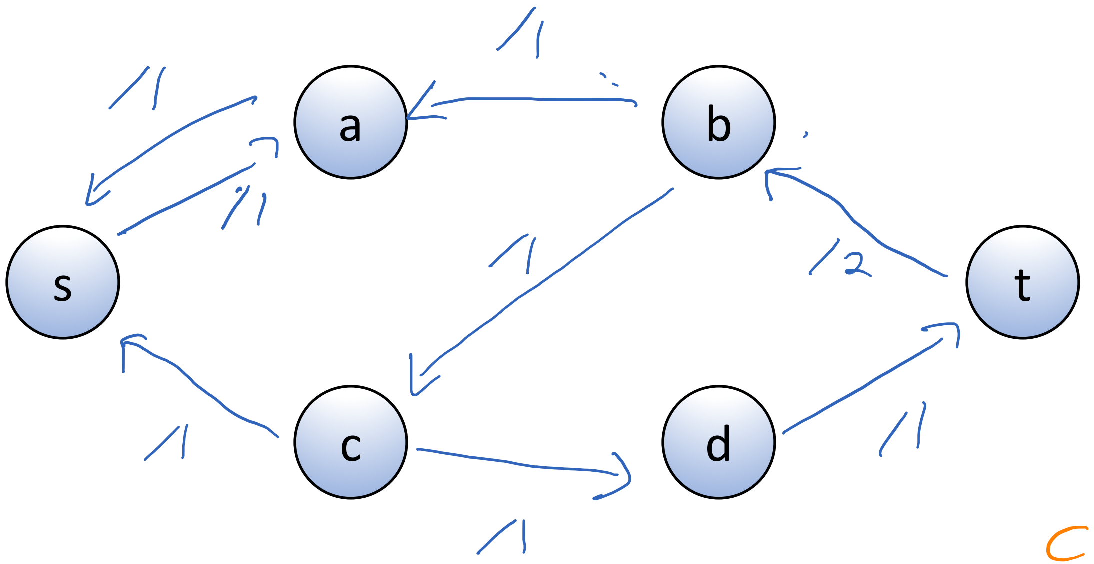


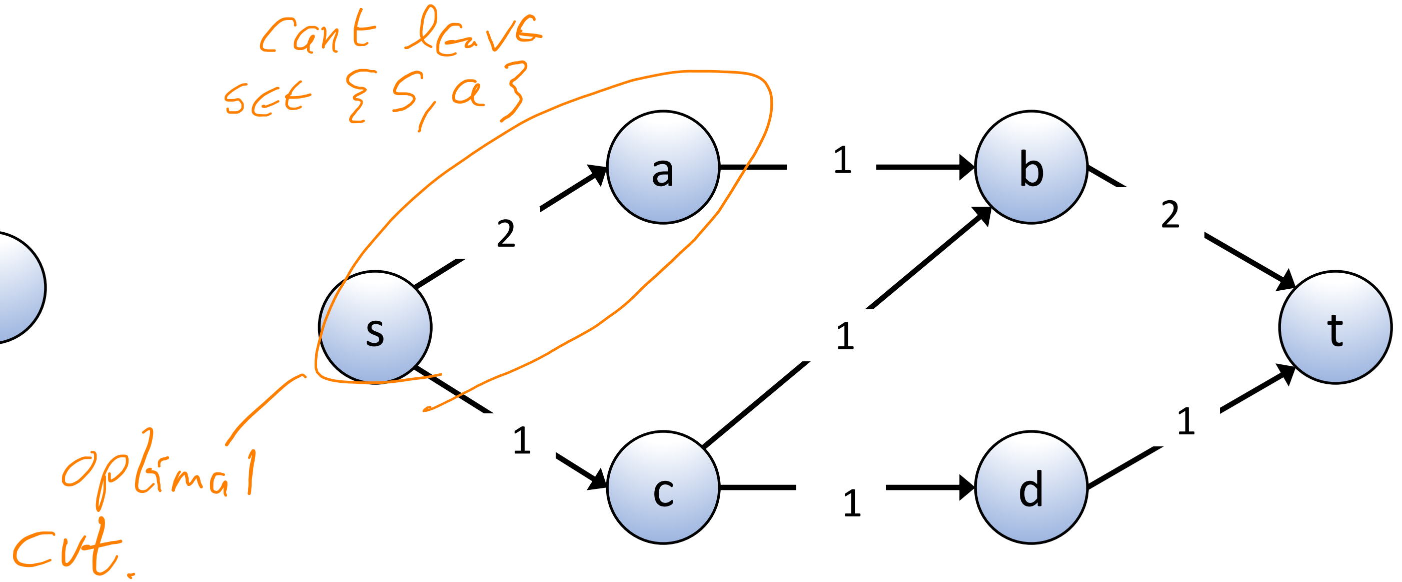
Ford-Fulkerson Demo

- Run Ford-Fulkerson on the following network



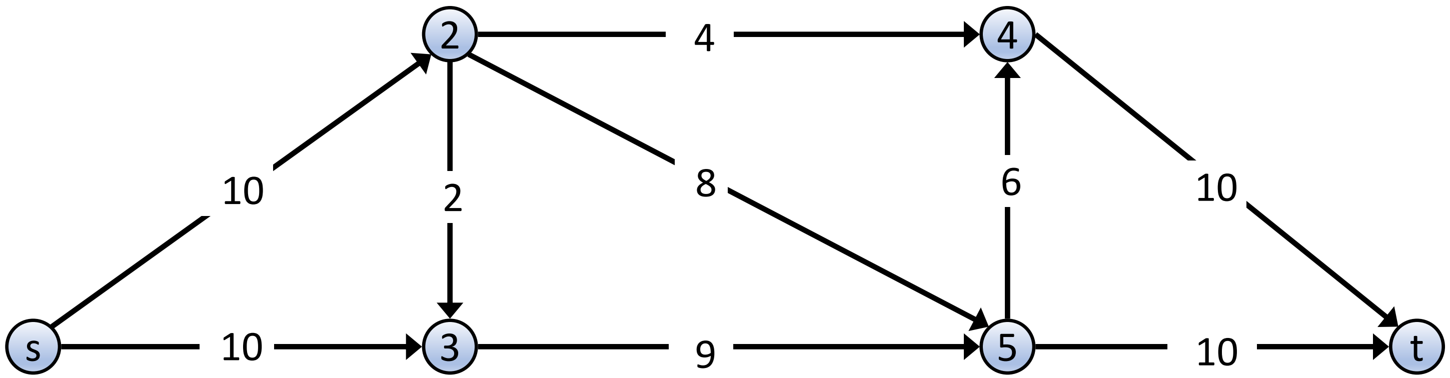






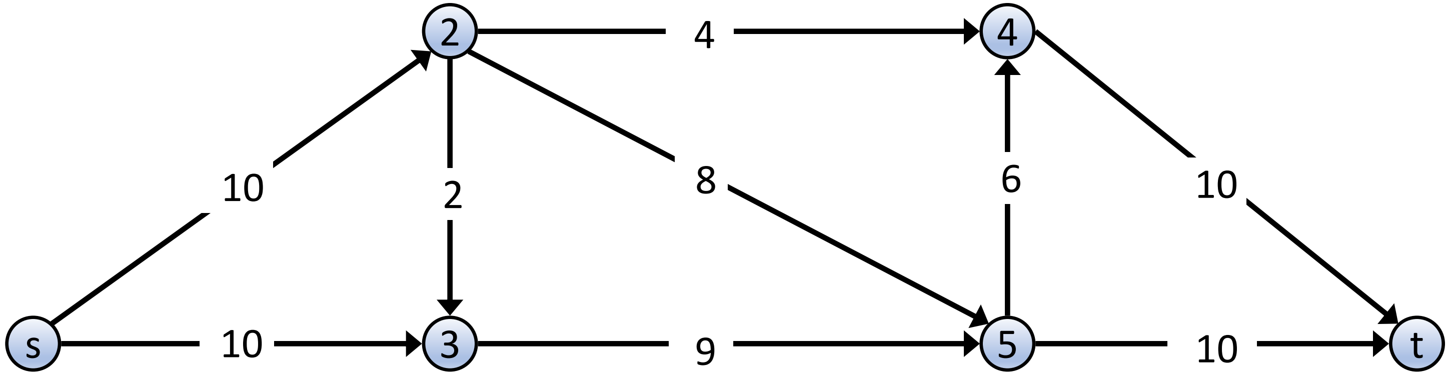
Ford-Fulkerson Demo

G:



Ford-Fulkerson Demo

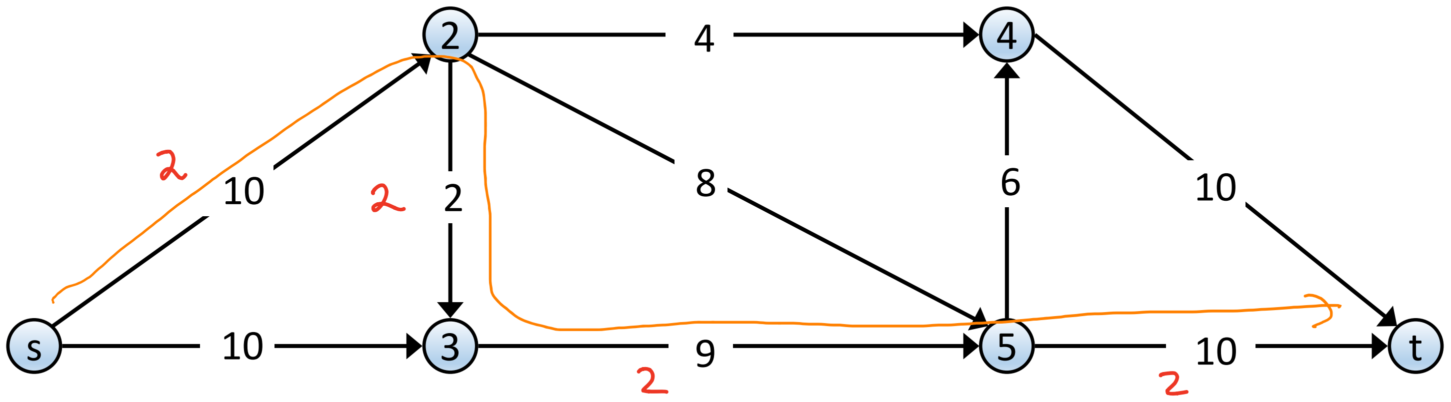
G:



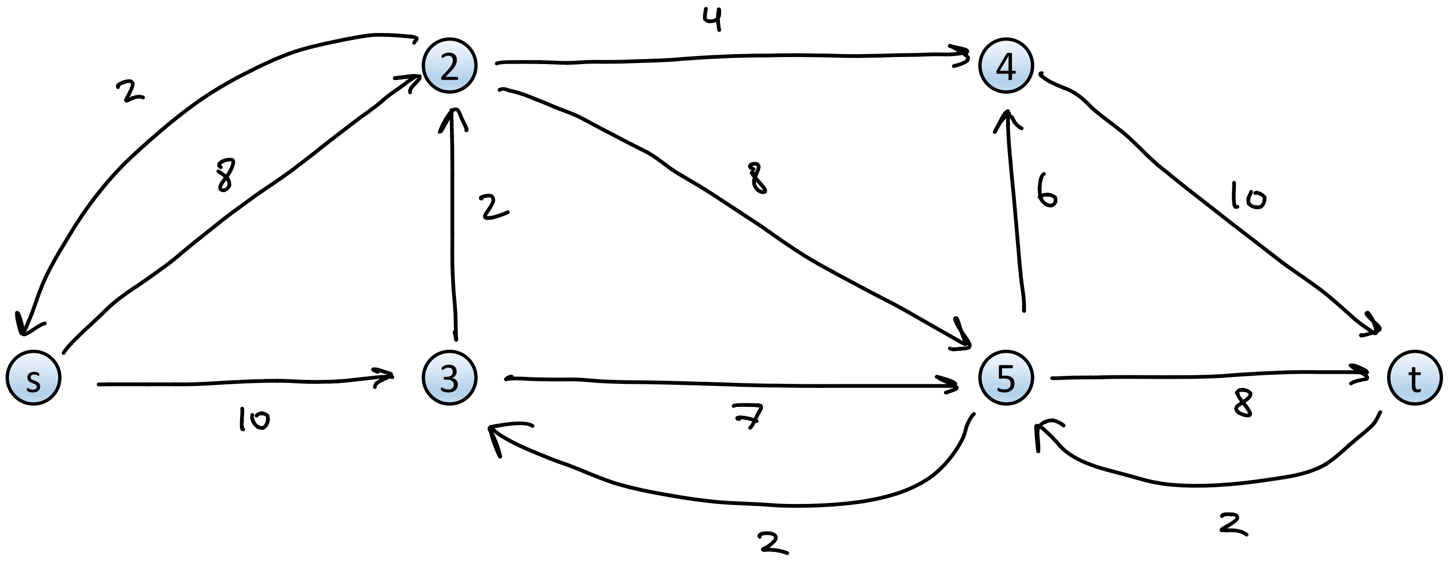
G_f:

Ford-Fulkerson Demo

G:

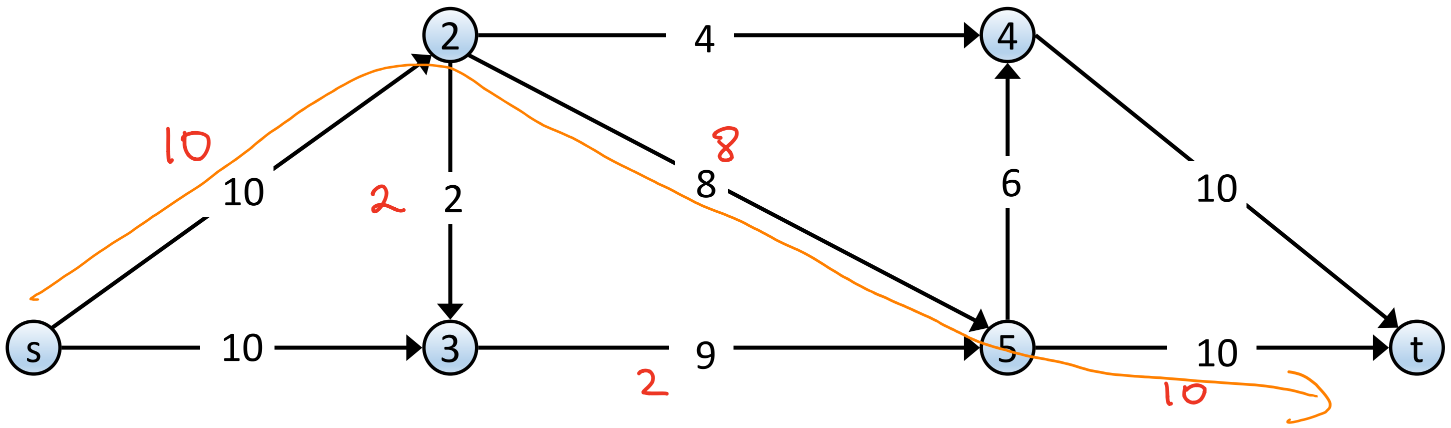


G_f:

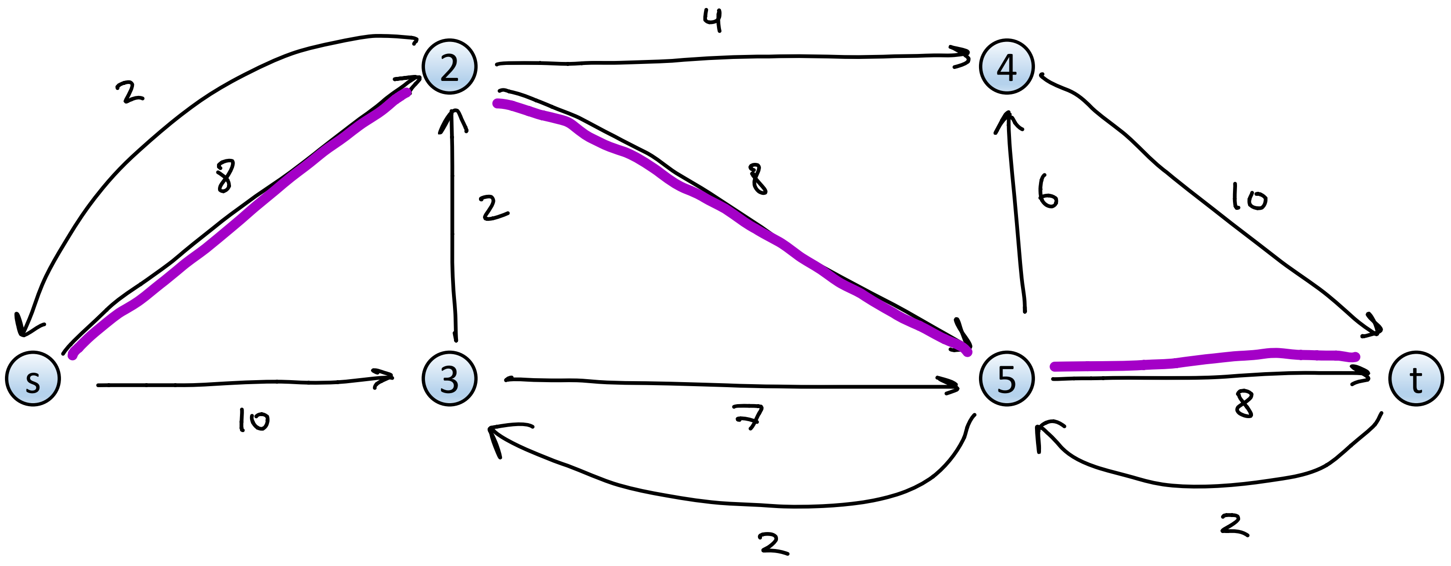


Ford-Fulkerson Demo

G:

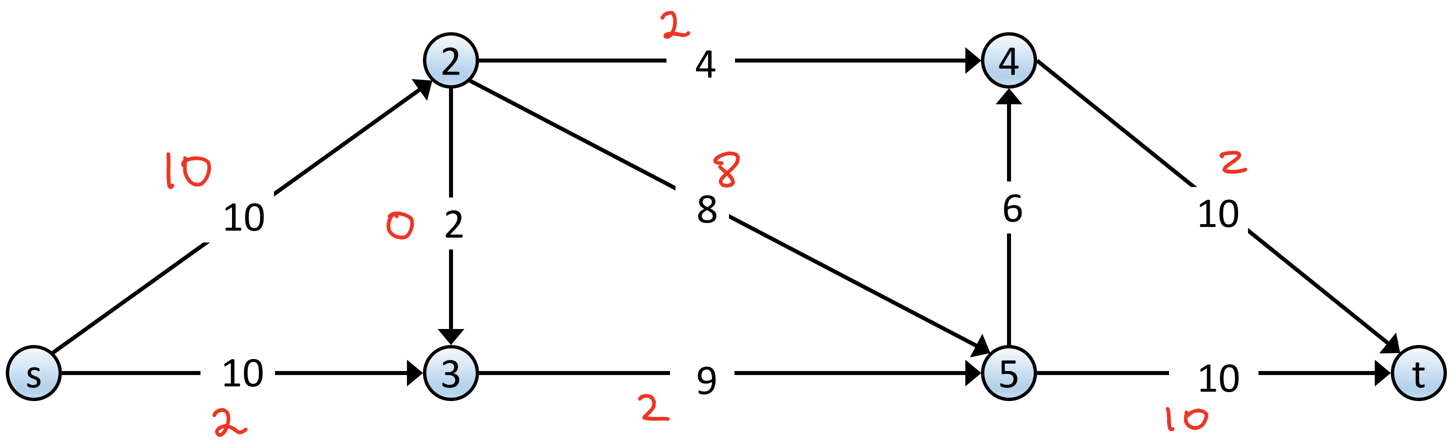


G_f:

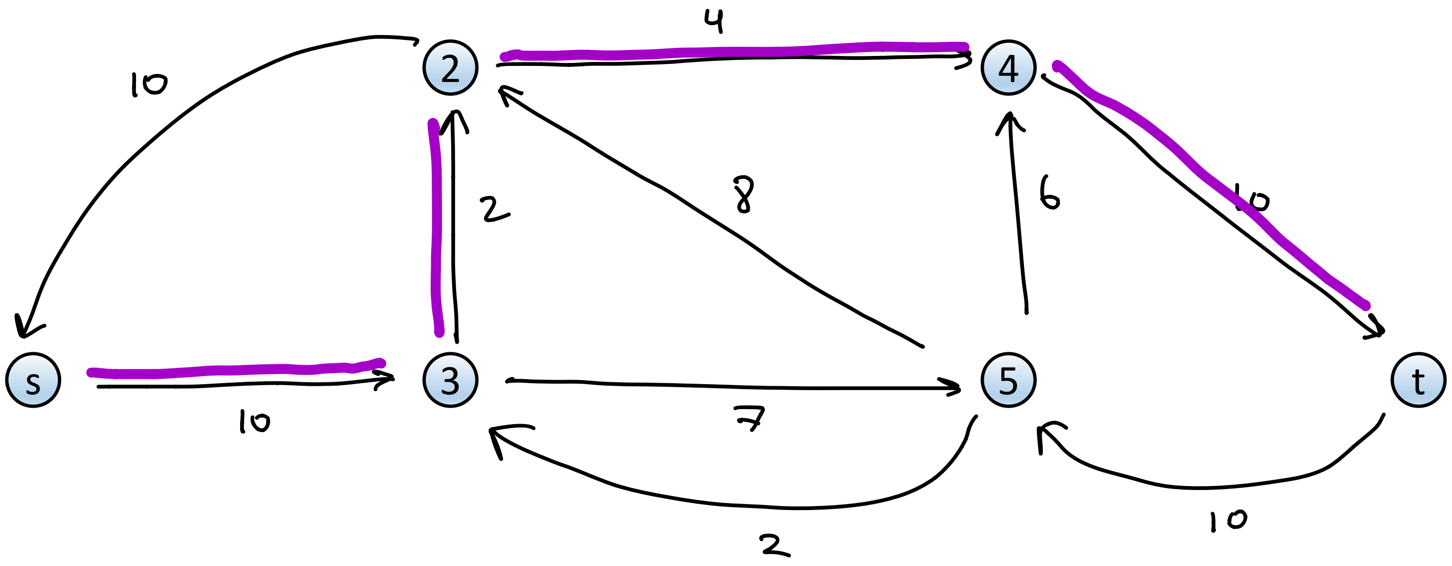


Ford-Fulkerson Demo

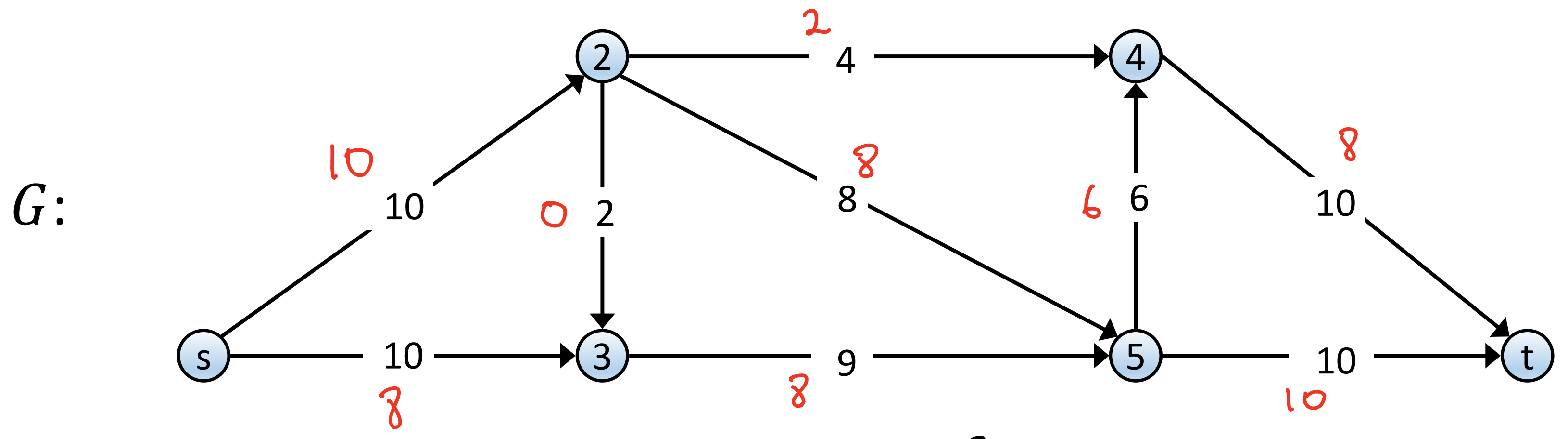
G:

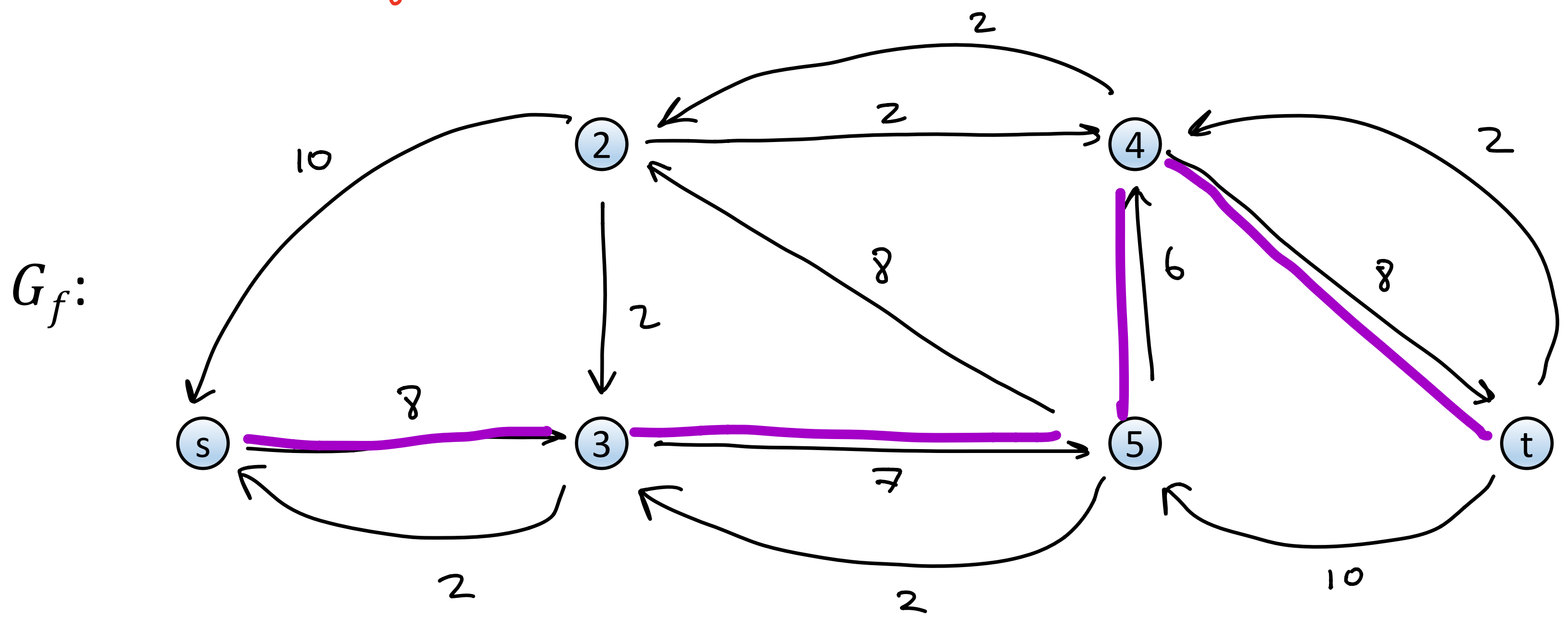


G_f:



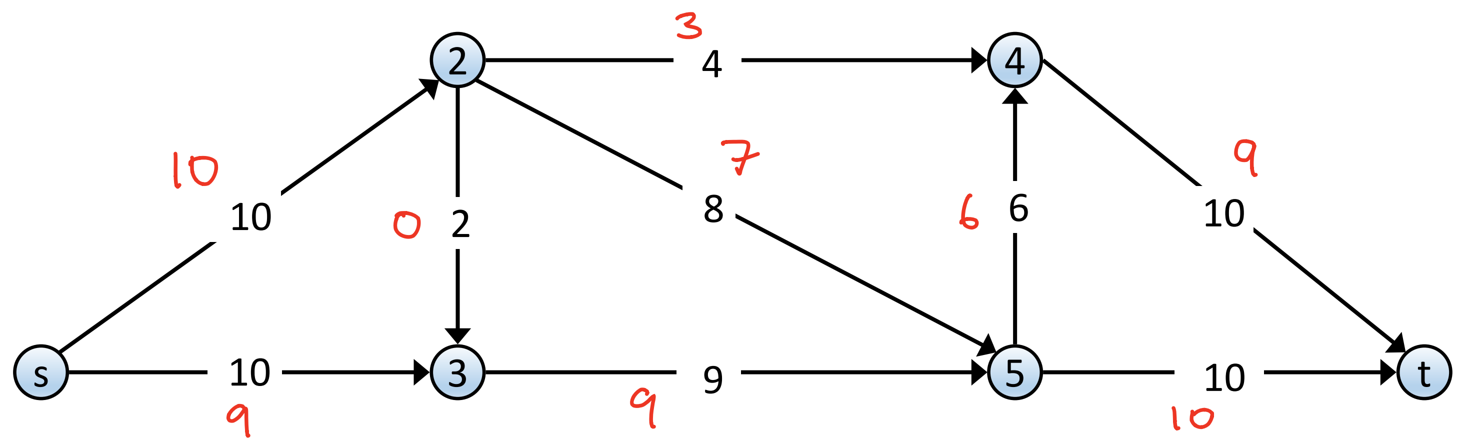
Ford-Fulkerson Demo



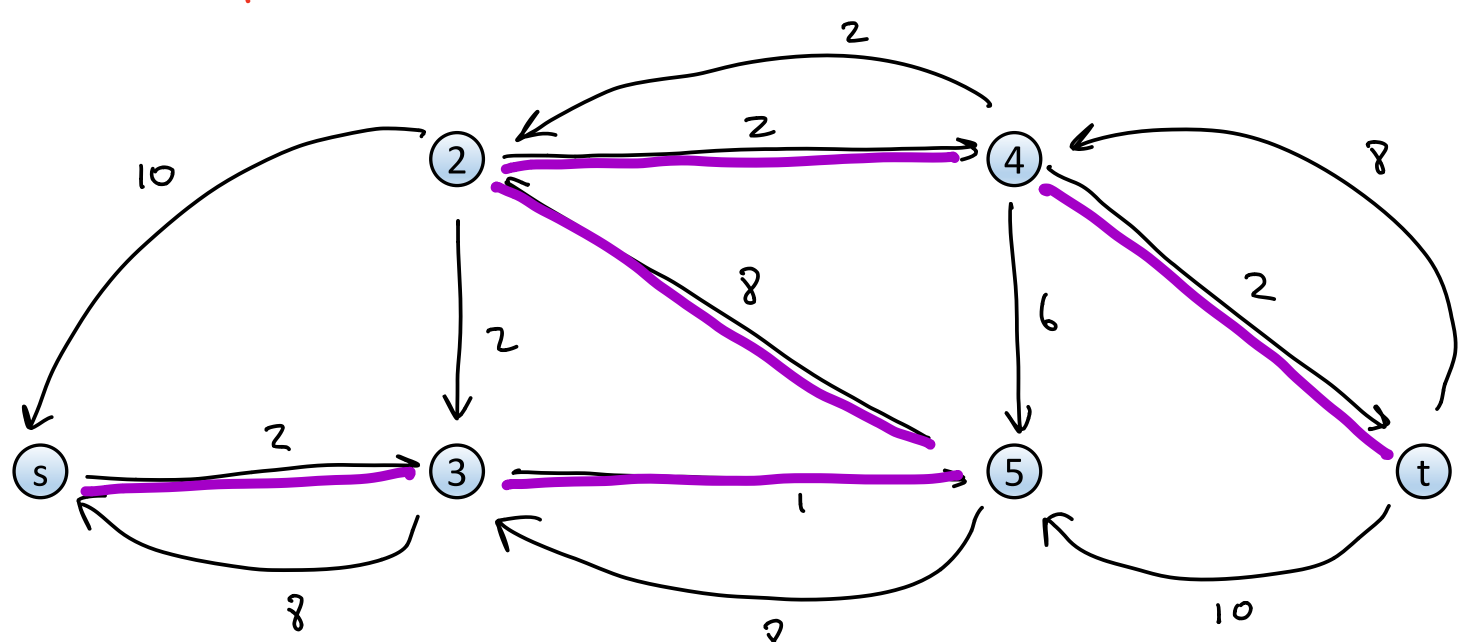


Ford-Fulkerson Demo

G:



G_f:



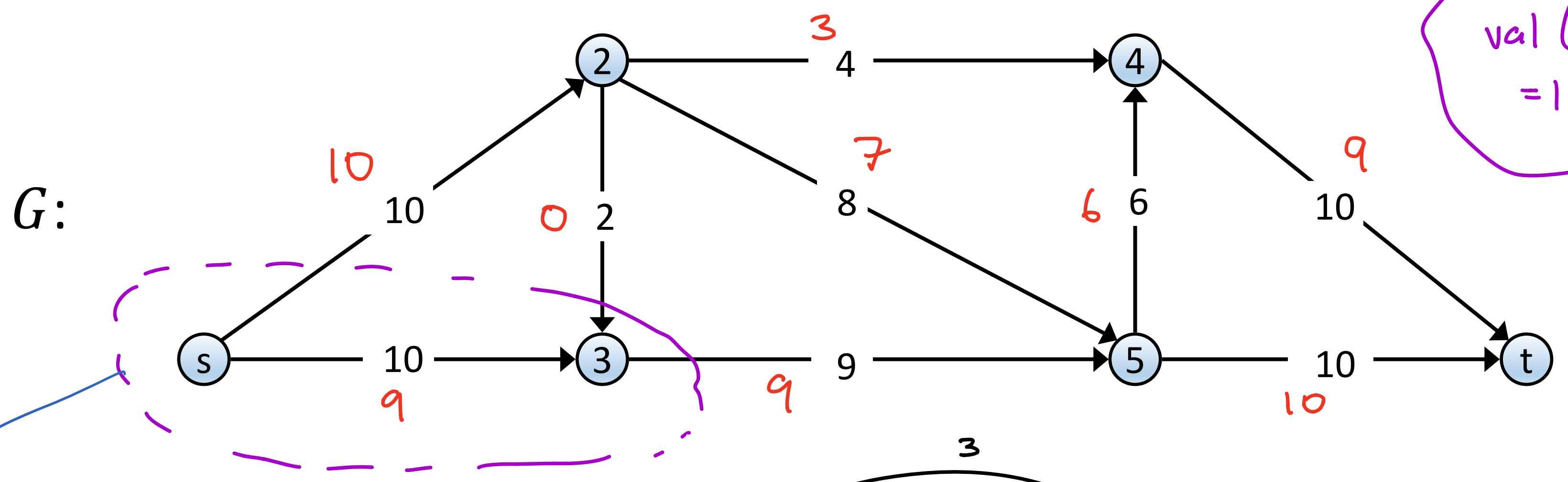
Ford-Fulkerson Demo

$A = \{s, 3\}$

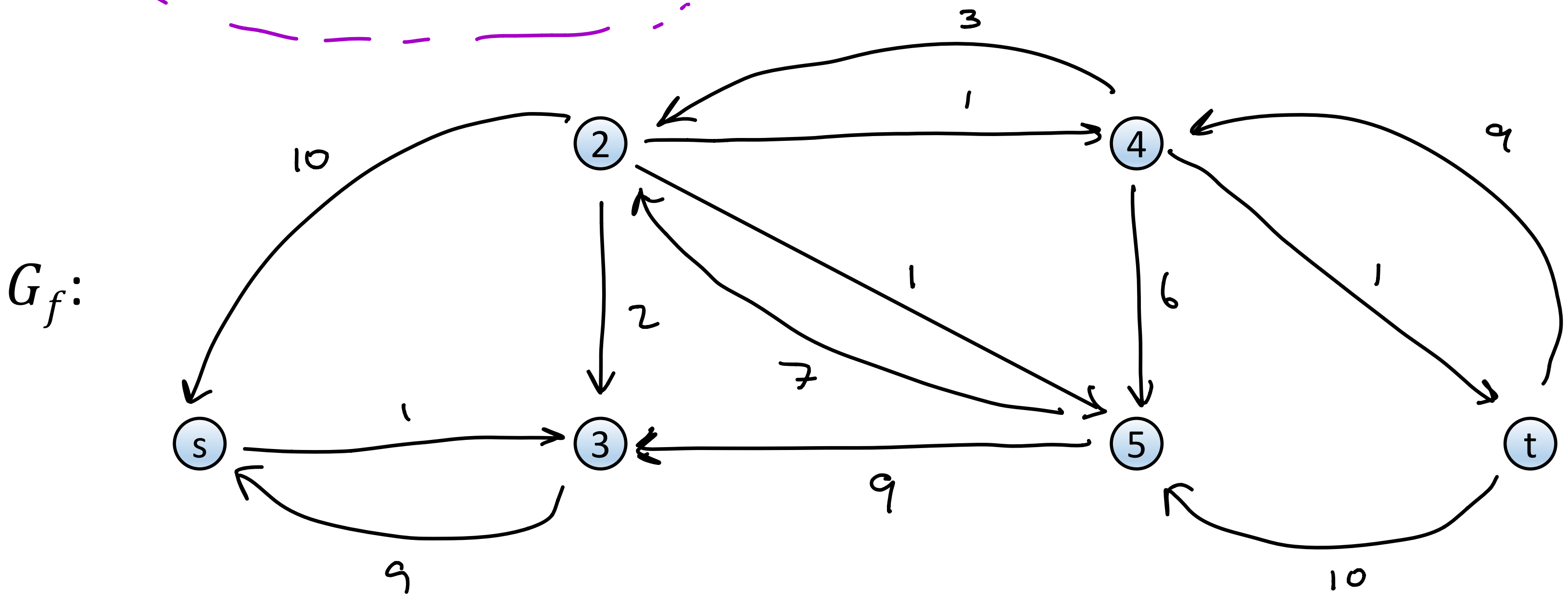
$B = \{2, 4, 5, t\}$

$\text{cap}(A,B) = 19$

$\text{val}(f) = 19$

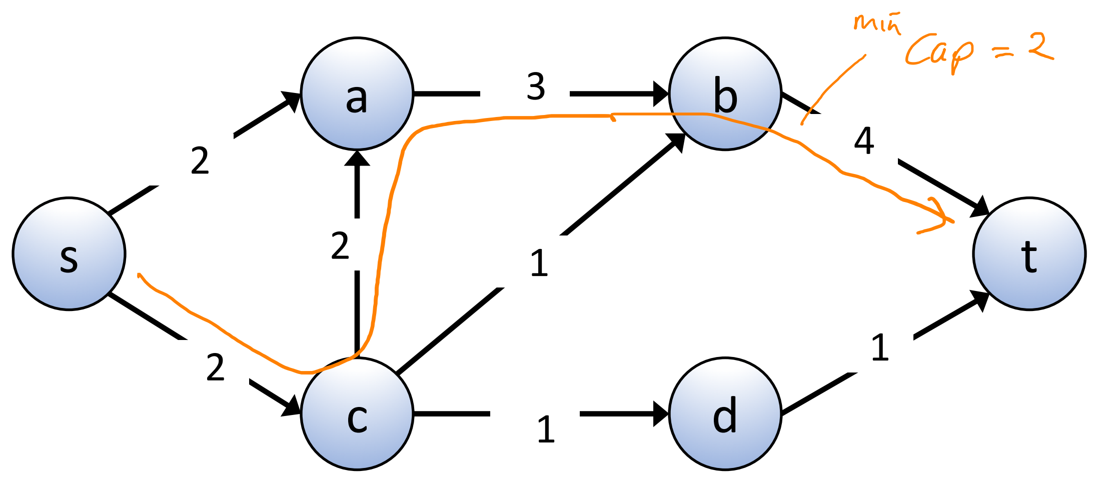


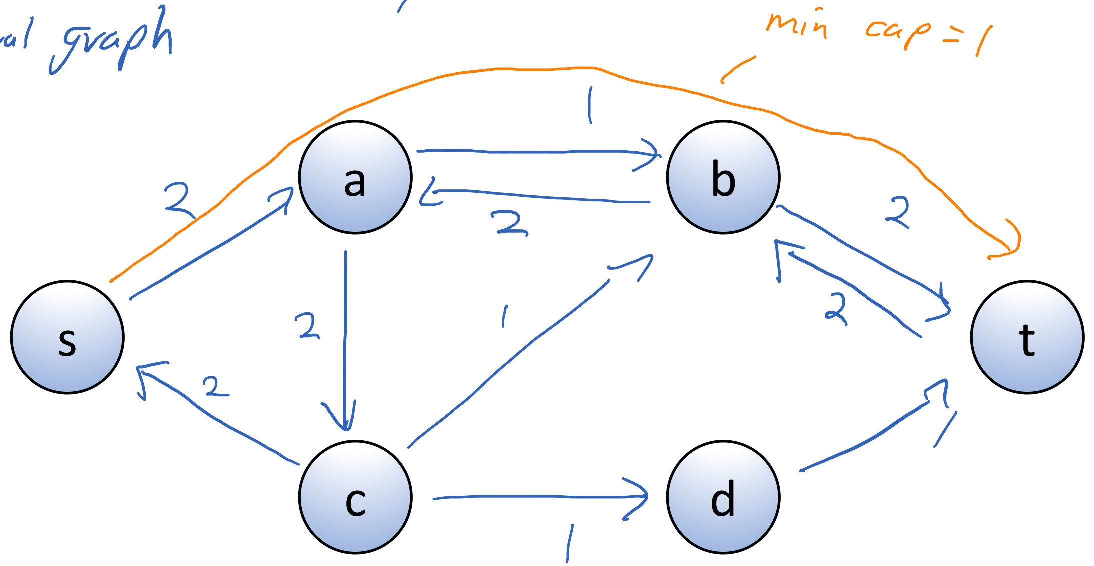
Optimal cut (nodes accessible to S under G_f)

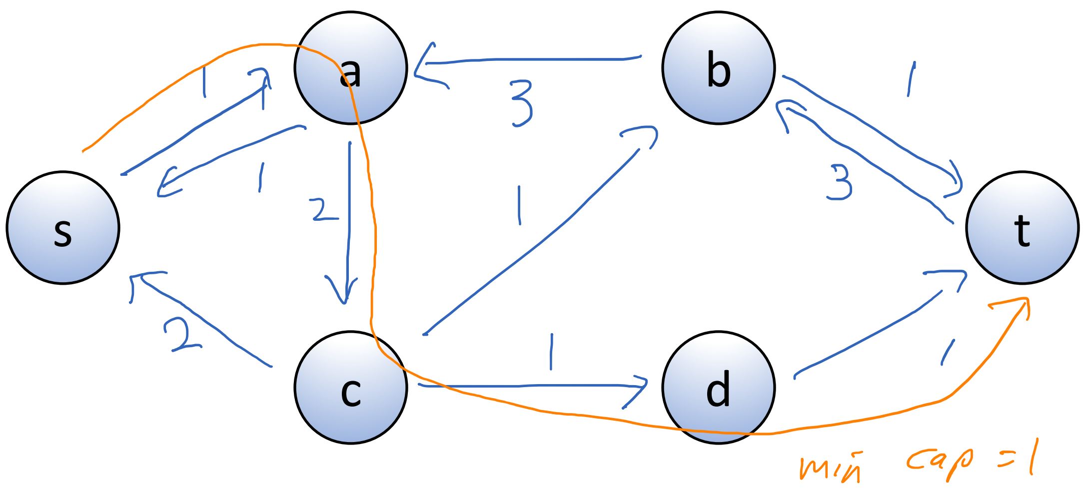


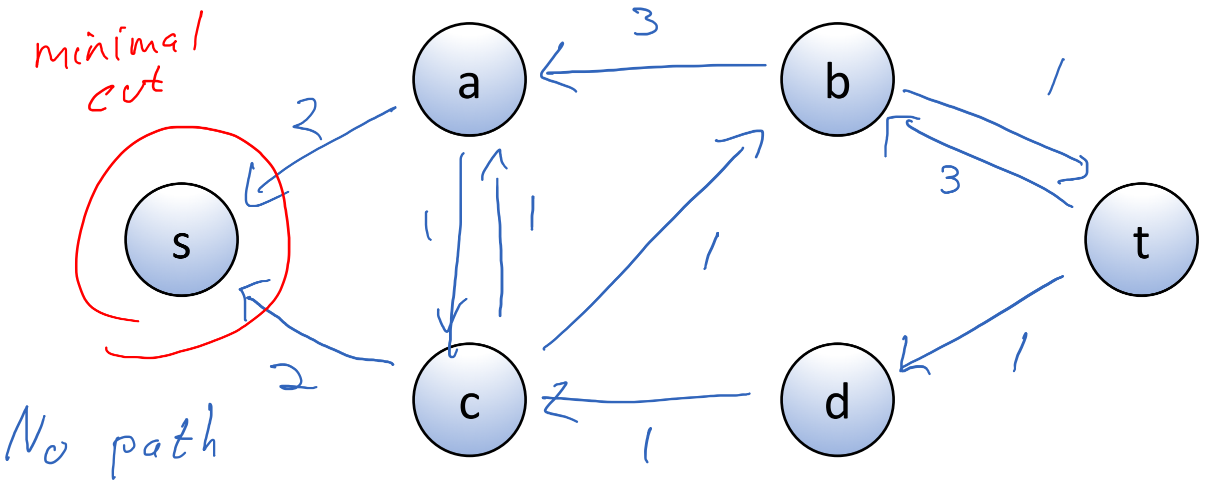
Ford-Fulkerson Demo

- Run Ford-Fulkerson on the following network
  - Find a path in residual graph, max it out
  - update Residual graph









What do we want to prove?

- FF Terminates
- FF finds a maximum s-t flow
- There is always a cut (A,B) such that val(f) = cap(A,B)

Ford-Fulkerson Algorithm – Run Time

```
FordFulkerson(G,s,t,{c})
    for e ∈ E: f(e) ← 0
    Gf is the residual graph

    while (there is an s-t path P in Gf)
        f ← Augment(Gf,P)
        update Gf

    return f
```

if all capacities are integers, each loop will achieve 1 unit more flow

Find a path $O(m)$

$O(n)$ (worst path has at most n nodes)

$O(n)$

```
Augment(Gf, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if e ∈ E:   f(e) ← f(e) + b
        else:       f(e) ← f(e) - b
    return f
```

$O(m) \times$ # paths selected

at most max value of flow

Run time: $m \times f_{max}$

Running Time of Ford-Fulkerson

f^* is maximal flow

- For **integer capacities**, $\leq val(f^*)$ augmentation steps

- Can perform each augmentation step in $O(m)$ time
  - find augmenting path in $O(m)$ — BFS, for example
  - augment the flow along path in $O(n)$
  - update the residual graph along the path in $O(n)$

- For integer capacities, FF runs in $O\big(m \cdot val(f^*)\big)$ time
  - $O(mn)$ time if all capacities are $c_e = 1$
  - $O(mnC_{\max})$ time for any integer capacities
  - Problematic when capacities are large

Can do better $O(mn)$ alg's exist

Correctness of Ford-Fulkerson

- **Theorem:** f is a maximum s-t flow if and only if there is no augmenting s-t path in G_f

- **(Strong) MaxFlow-MinCut Duality:** The value of the max s-t flow equals the capacity of the min s-t cut

- We'll prove that the following are equivalent for all f
  1. There exists a cut (A, B) such that $val(f) = cap(A, B)$
  2. Flow f is a maximum flow
  3. There is no augmenting path in G_f

$1 \Rightarrow 2$

$2 \Rightarrow 3$

$3 \Rightarrow 1$

Optimality of Ford-Fulkerson

- **Theorem:** the following are equivalent for all f
  1. There exists a cut (A, B) such that $val(f) = cap(A, B)$
  2. Flow f is a maximum flow
  3. There is no augmenting path in G_f

$1 \Rightarrow 2$ (weak duality)

value of any flow was $\leq$ capacity of any cut

$2 \Rightarrow 3$

If ~~th~~ there is an augmenting path, could send flow down it, increasing $val(f)$ X

$3 \Rightarrow 1$ is more challenging

Optimality of Ford-Fulkerson

- **(3 → 1)** If there is no augmenting path in G_f, then there is a cut (A, B) such that $val(f) = cap(A, B)$
  - Let A be the set of nodes reachable from s in G_f
  - Let B be all other nodes

Need to show: (A,B) is a cut

This is true b/c sink t is not in A.

If t were in A, there would be a path from s → t in G_f, which is an augmenting path. ✗

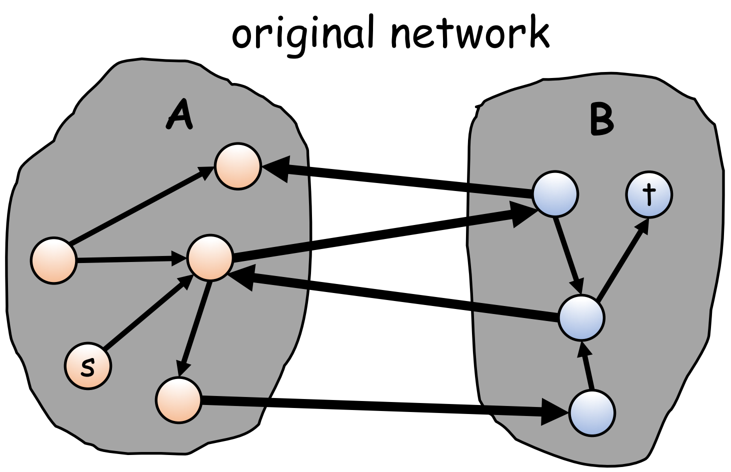
Need to show $cap(A,B) = val(f)$

Optimality of Ford-Fulkerson

- **(3 → 1)** If there is no augmenting path in G_f, then there is a cut (A, B) such that $val(f) = cap(A, B)$
  - Let A be the set of nodes reachable from s in G_f
  - Let B be all other nodes
  - **Key observation:** no edges in G_f go from A to B

e is in original graph

- If e is $A \to B$, then $f(e) = c(e)$
- If e is $B \to A$, then $f(e) = 0$



$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= \sum_{e \text{ out of } A} c(e) - 0 = Cap(A, B)$$

Summary

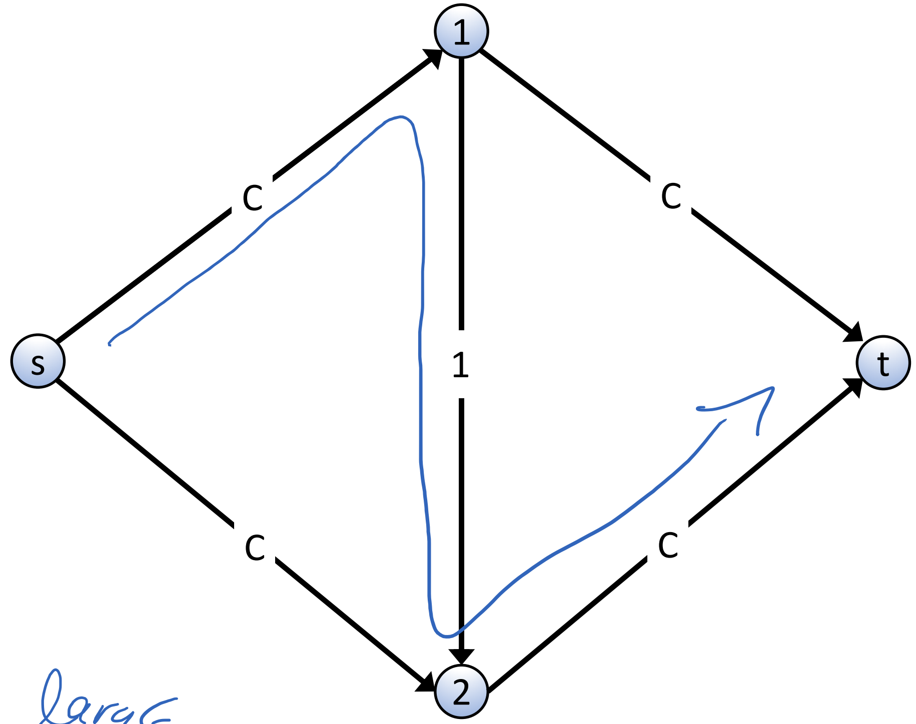
- **The Ford-Fulkerson Algorithm solves maximum s-t flow**
  - Running time $O\big(m \cdot val(f^*)\big)$ in networks with integer capacities
  - Space $O(n+m)$

- **MaxFlow-MinCut Duality:** The value of the maximum s-t flow equals the capacity of the minimum s-t cut
  - If f^* is a maximum s-t flow, then the set of nodes reachable from s in G_{f^*} gives a minimum cut
  - Given a max-flow, can find a min-cut in time $O(n+m)$

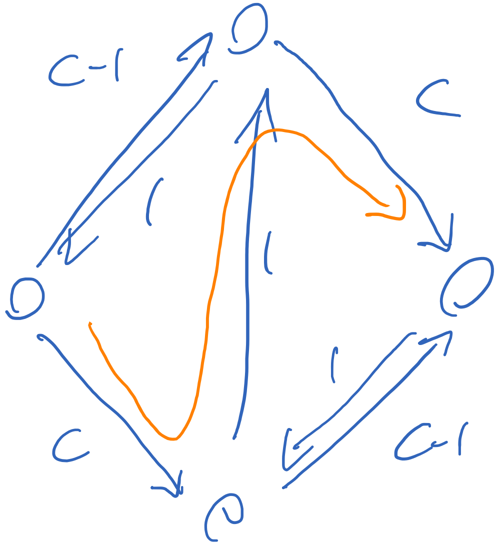
- **Every graph with integer capacities has an integer maximum flow**
  - Ford-Fulkerson will return an integer maximum flow

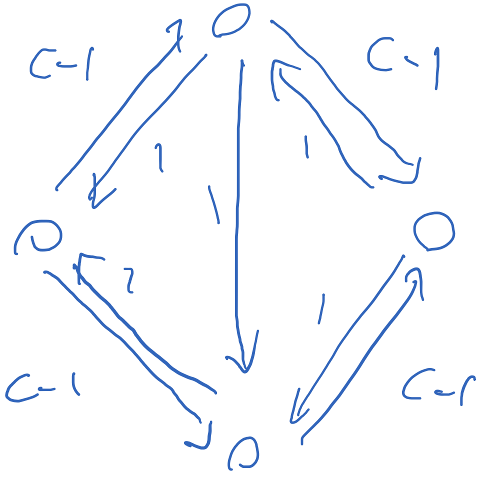
Ford-Fulkerson Algorithm is slow if augmenting paths are not chosen well

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck



C large





Repeat 2 · C times
Very slow

Choosing Good Augmenting Paths

- If augmenting paths are chosen arbitrarily:
  - If FF terminates, it outputs a maximum flow
  - Might not terminate, or might require many augmentations

- Augmenting paths can be chosen cleverly
  - Maximum-capacity augmenting path ("fattest augmenting path")
  - Shortest augmenting paths ("shortest augmenting path")

keep track of bottleneck in BFS

BFS