CS3000: Algorithms & Data
Paul Hand

Lecture 17:

- Implementation of Dijkstra's Algorithm (Data Structures)

Mar 20, 2019

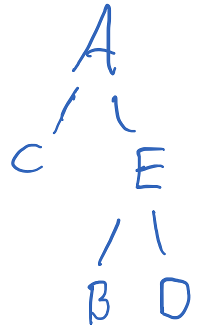
Execute Dijkstra's Algorithm: Activity
Find distances from A and the shortest path tree

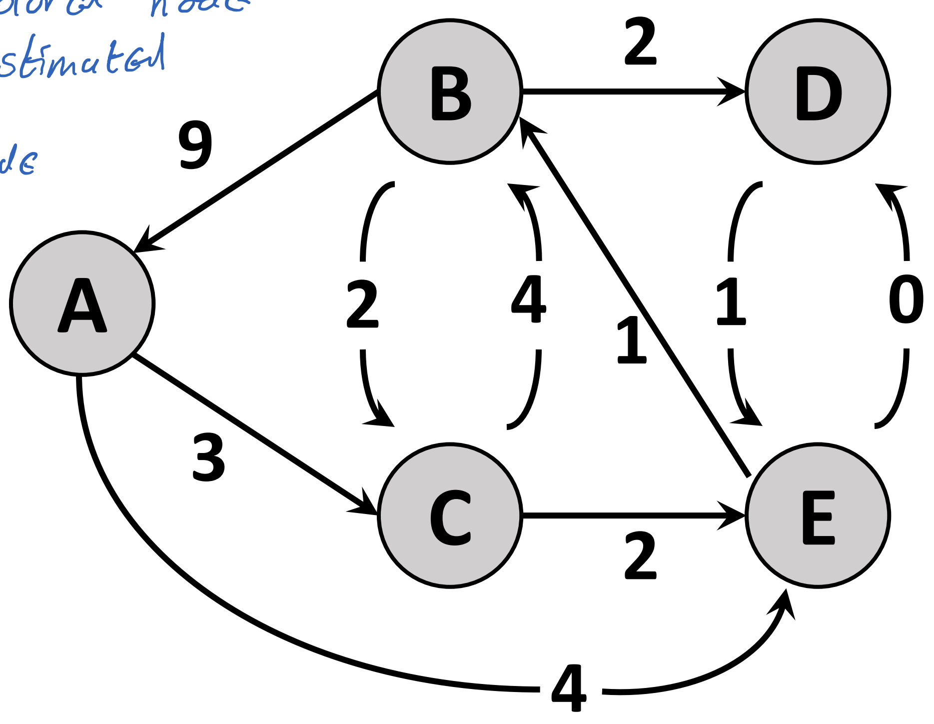
	A	B	C	D	E
$d_0(u)$	0	∞	∞	∞	∞
$d_1(u)$	0	∞	3	∞	4
$d_2(u)$	0	7	3	∞	4
$d_3(u)$	0	5	3	4	4
$d_4(u)$	0	5	3	4	4
$d_5(u)$	0		3		4

Look for unexplored node with shortest estimated distance

"Explore" that node by updating the best known distance to its neighbors.

Repeat





Implementing Dijkstra

n nodes
m edges

If Q is stored as an array, how much work?
n

```
Dijkstra(G = (V,E,{ℓ(e)}, s):
  d[s] ← 0, d[u] ←∞ for every u != s
  parent[u] ←⊥ for every u
  Q ← V                  // Q holds the unexplored nodes

  While (Q is not empty):
```
$u \leftarrow \operatorname{argmin}_{w \in Q} d[w]$ //Find closest unexplored
```
    Remove u from Q

    // Update the neighbors of u
    For ((u,v) in E):
      If (d[v] > d[u] + ℓ(u,v)):
        d[v] ←d[u] + ℓ(u,v)
        parent[v] ←u

  Return (d, parent)
```

How long are the steps (below in arrows)?

If we store d as an array, ~n work

current estimates

Value of w which minimizes d[w]

two lookups
sum
comparison

O(1)

Priority Queues / Heaps

Priority Queues

- Need a data structure Q to hold key-value pairs

key : node, u
value : d[u]

- Need to support the following operations
  - Insert(Q,k,v): add a new key-value pair
  - Lookup(Q,k): return the value of some key
  - ExtractMin(Q): identify the key with the smallest value
  - DecreaseKey(Q,k,v): reduce the value of some key

Priority Queues

- **Naïve approach:** Sorted List (by value)

Key	a	c	e	h	b	g	k	d	f
Value	1	3	5	8	10	20	42	45	50

- Activity: With n total items, how long would it take to perform
  - Insert(Q,k,v): add a new key-value pair? log n to find position at which it belongs + shift over everything in $\boxed{O(n)}$
  - Lookup(Q,k): return the value of some key? worst case O(n) – linear search if you dont store mapping of nodes -> index
  - ExtractMin(Q): identify the key with the smallest value?
  - DecreaseKey(Q,k,v): reduce the value of some key?

O(n)

O(n) shifting items 2-n

If you do store mapping node -> index, O(1) time

Priority Queues

- **Naïve approach:** linked lists (unsorted)

Key	a	c	e	h	b	g	k	d	f
Value	11	12	2	36	4	20	42	10	8

- Activity: With n total items, how long would it take to perform
  - Insert(Q,k,v): add a new key-value pair? $O(1)$ to put at beginning
  - Lookup(Q,k): return the value of some key? $O(n)$ w/o mapping $O(1)$ with map
  - ExtractMin(Q): identify the key with the smallest value? $O(n)$
  - DecreaseKey(Q,k,v): reduce the value of some key? $O(1)$ w/ mapping

Priority Queues

- **Naïve approach:** linked lists

Key	a	c	e	h	b	g	k	d	f
Value	11	12	2	36	4	20	42	10	8

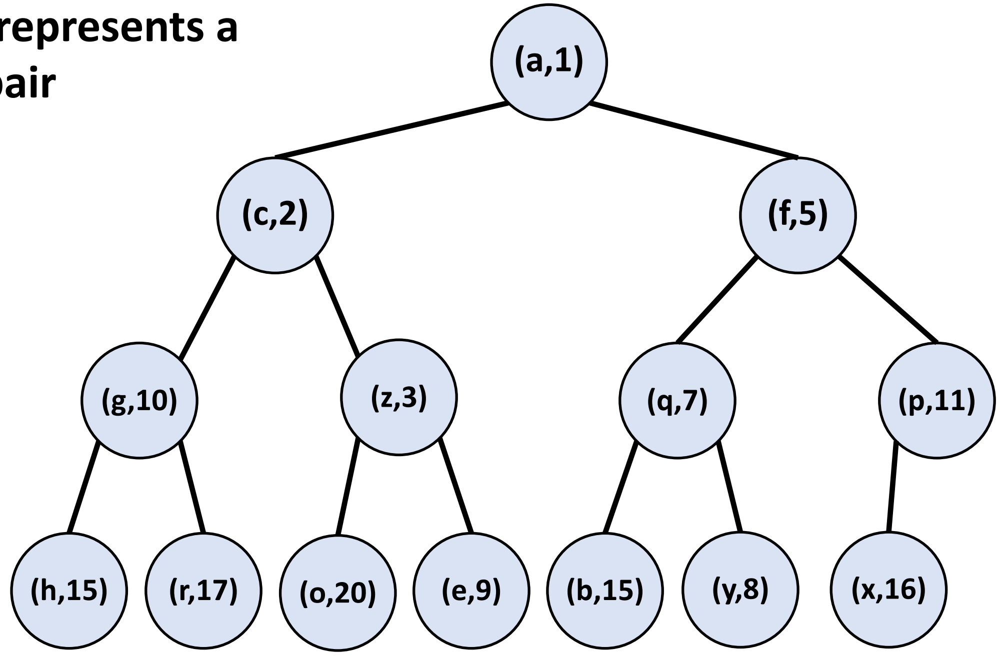
  - Insert takes O(1) time
  - ExtractMin, DecreaseKey take O(n) time

- **Binary Heaps:** implement all operations in O(log n) time where n is the number of keys

Heaps

- **Organize key-value pairs as a binary tree**
  - Later we'll see how to store pairs in an array
- **Heap Order:** If a is the parent of b, then v(a) ≤ v(b)

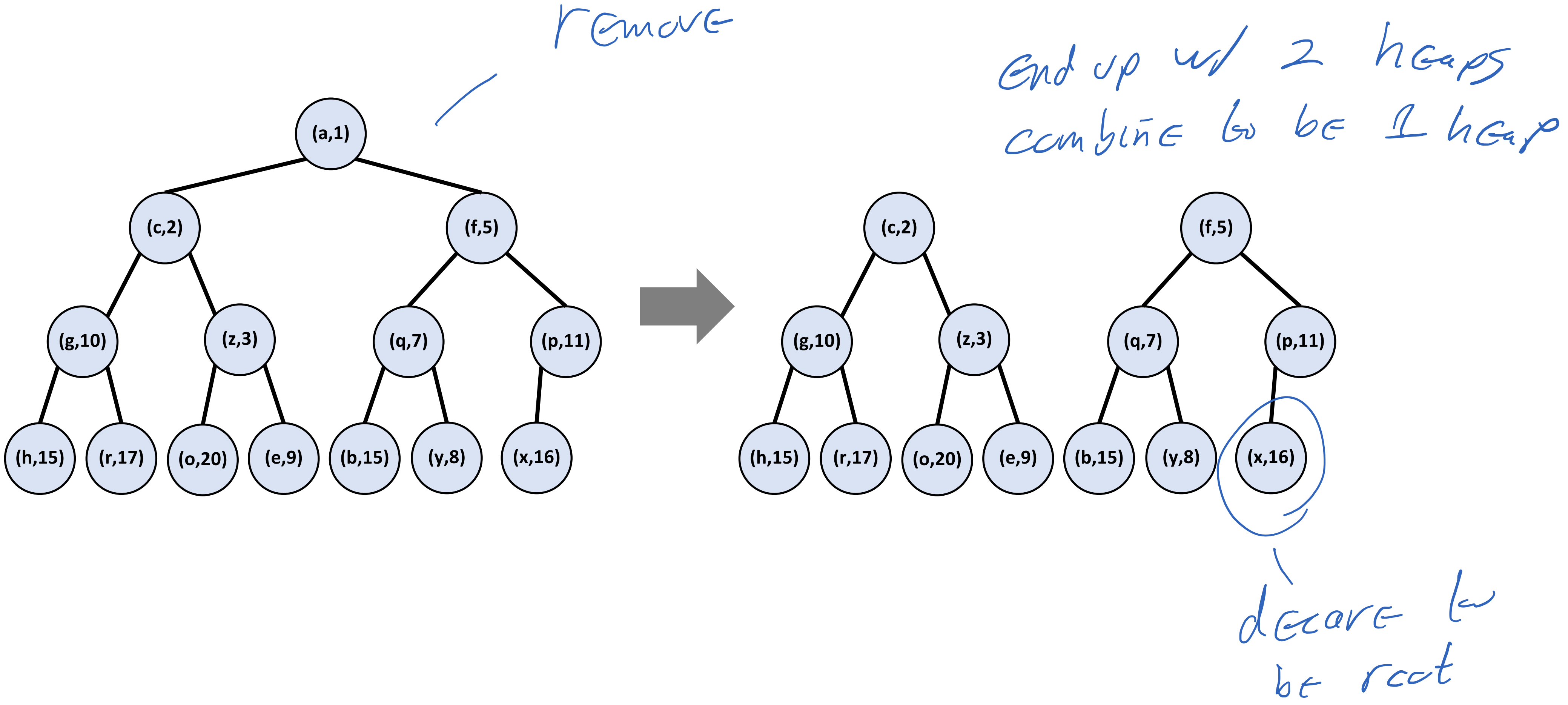
Each node represents a key-value pair



Ordering only along tree
not across tree

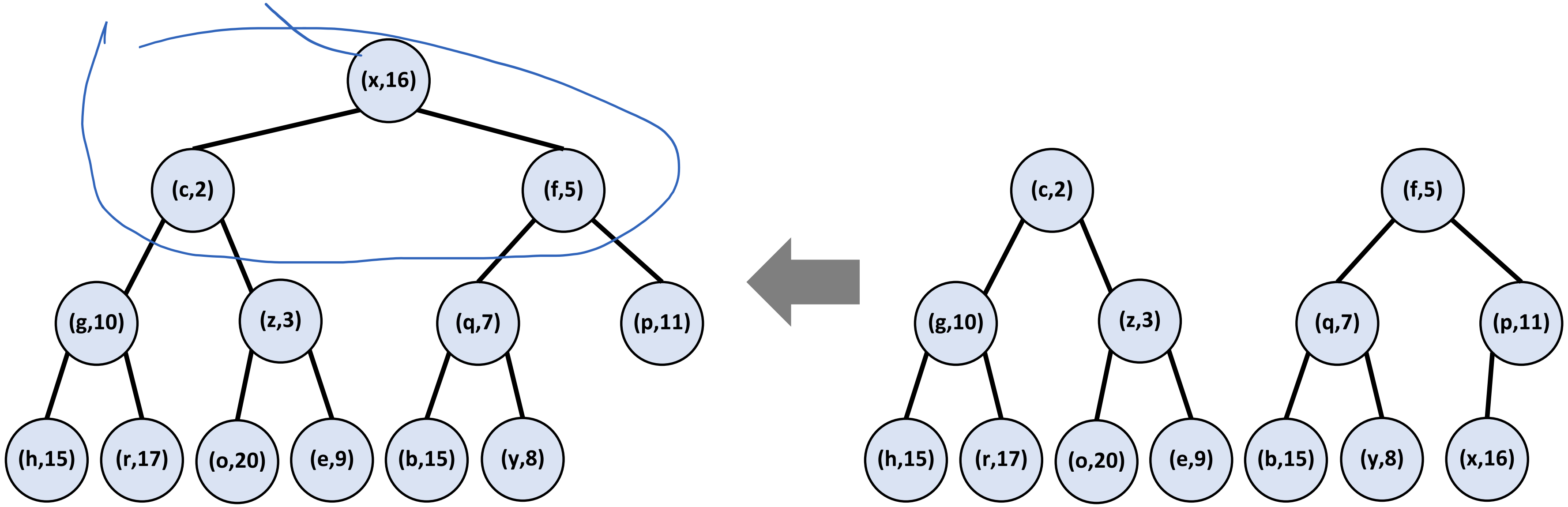
Not Sorted

Implementing ExtractMin

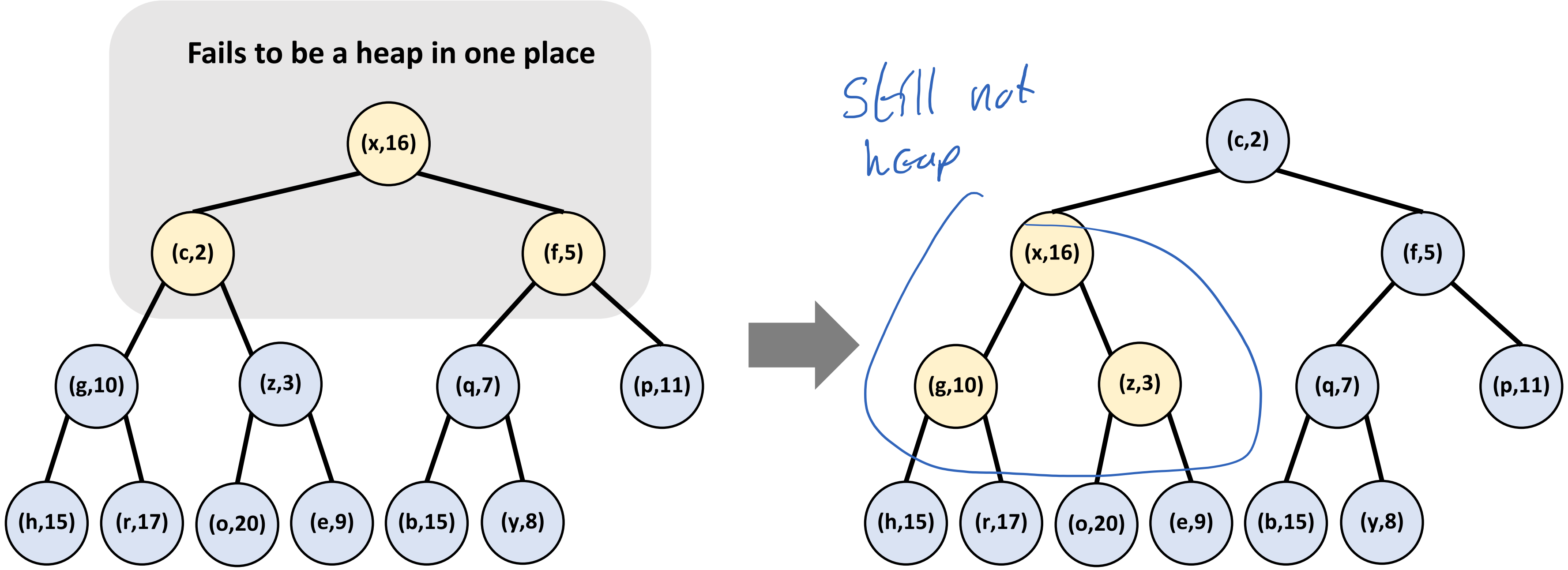


Implementing ExtractMin

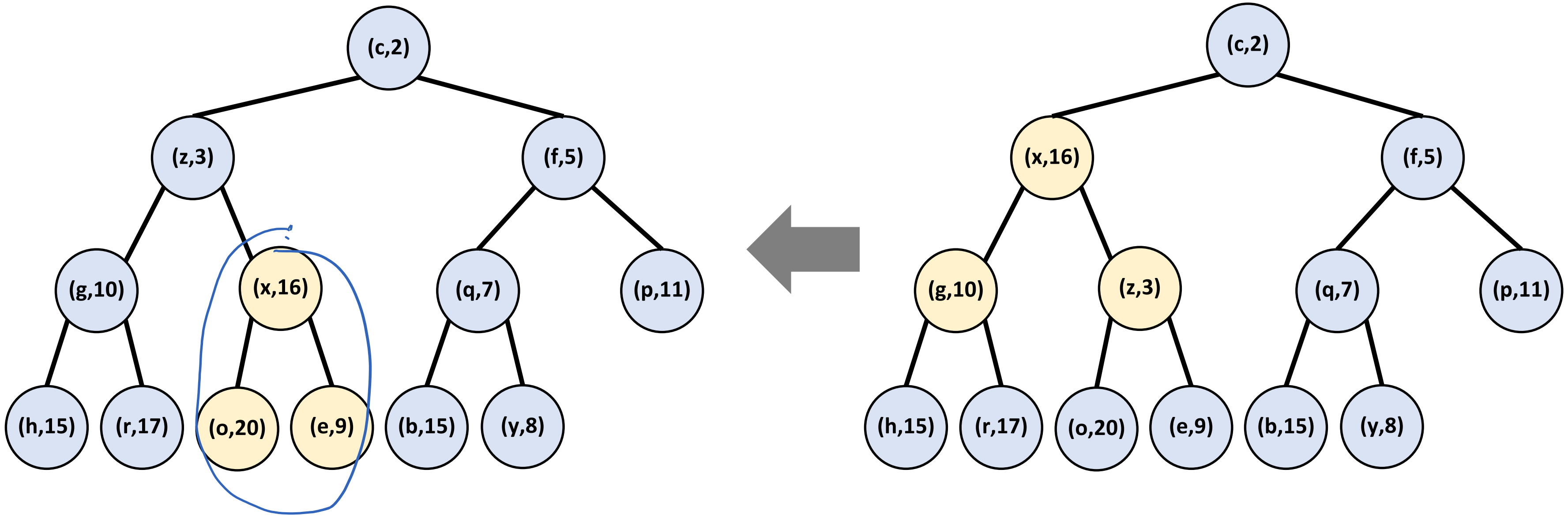
this violates heap order



Implementing ExtractMin

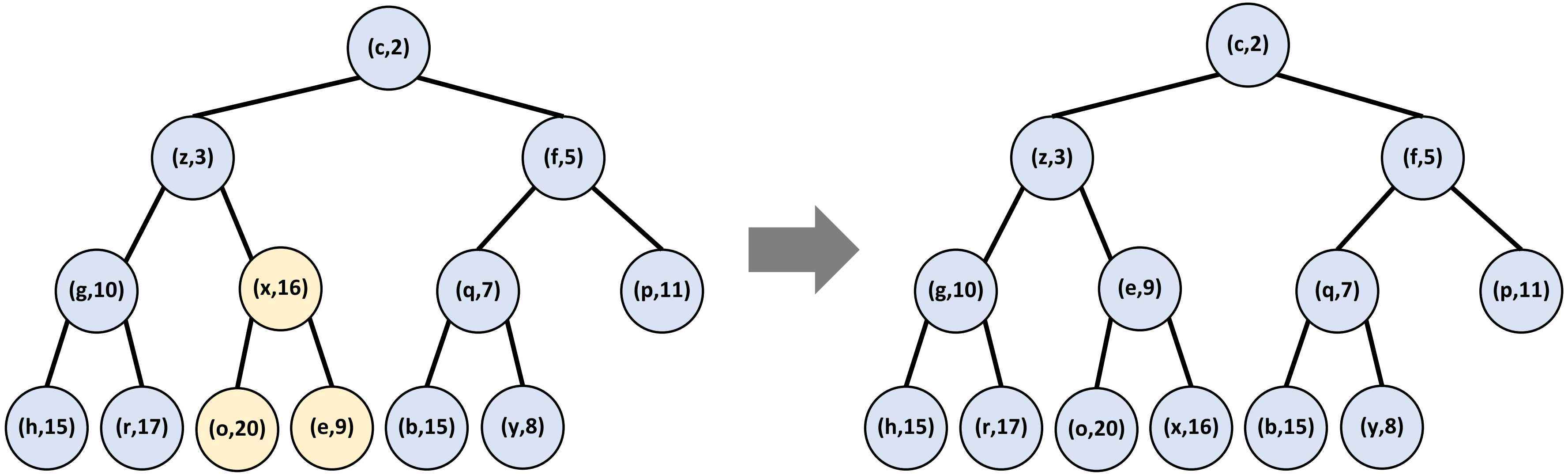


Implementing ExtractMin



Still
not heap

Implementing ExtractMin

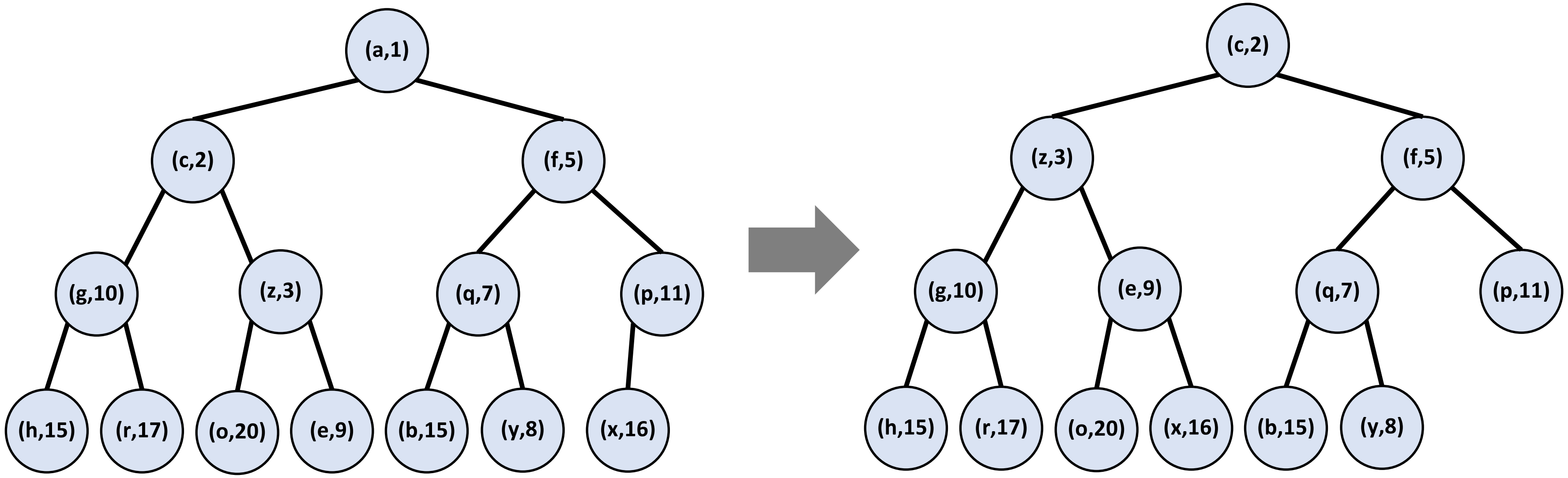


now its
a heap

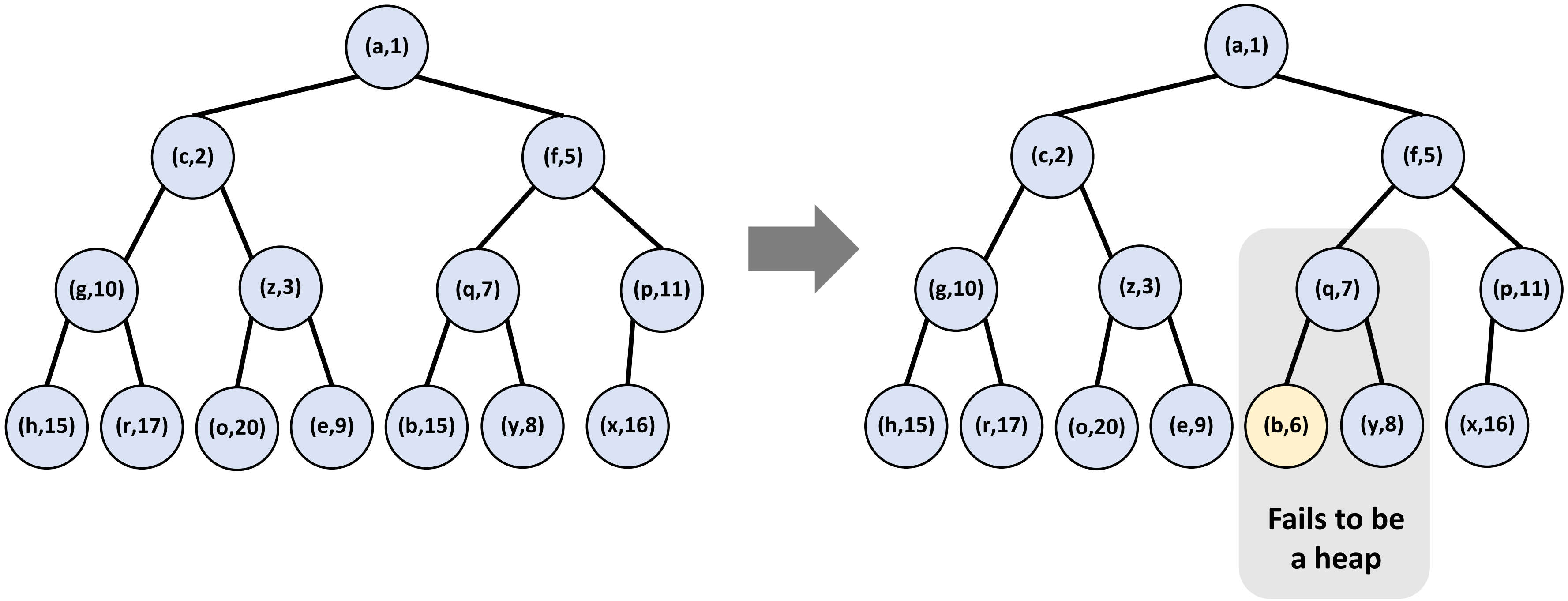
Implementing ExtractMin

- Three steps:
  - Pull the minimum from the root $O(1)$
  - Move the last element to the root $O(1)$
  - Repair the heap-order (heapify down) $O(\log n)$

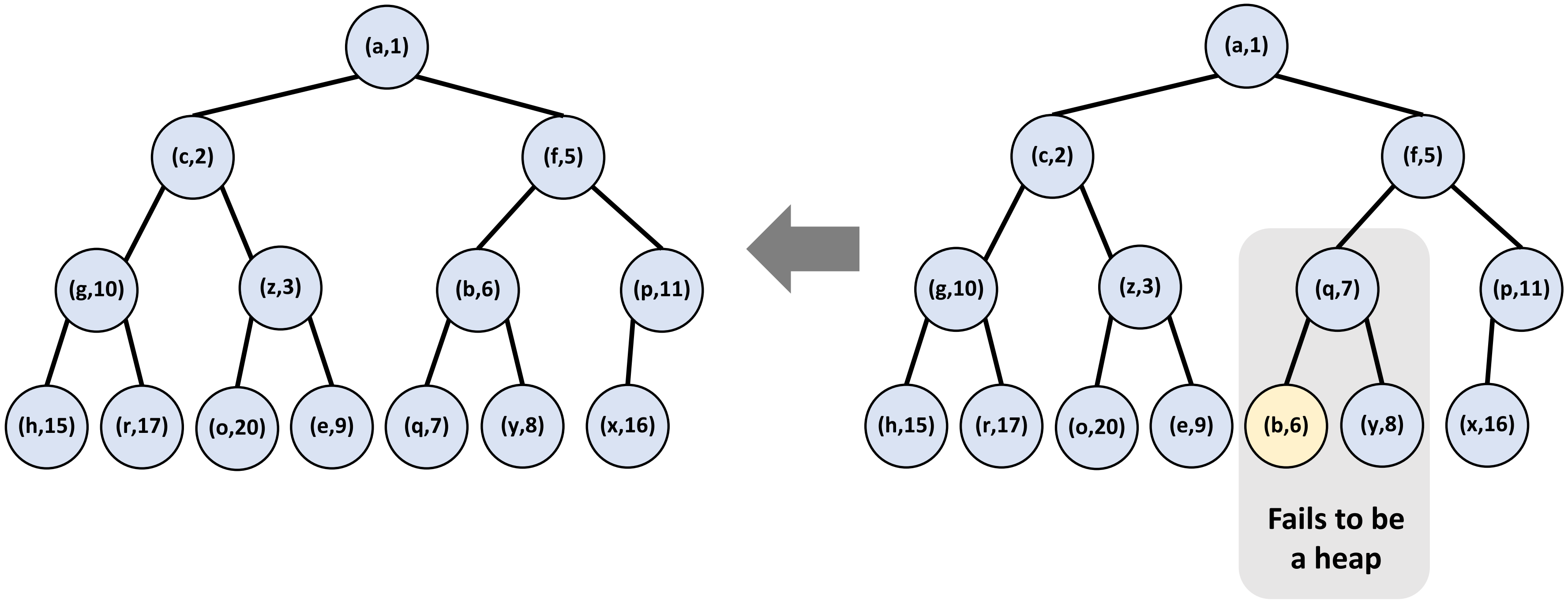
$\Rightarrow O(\log n)$



Implementing DecreaseKey



Implementing DecreaseKey

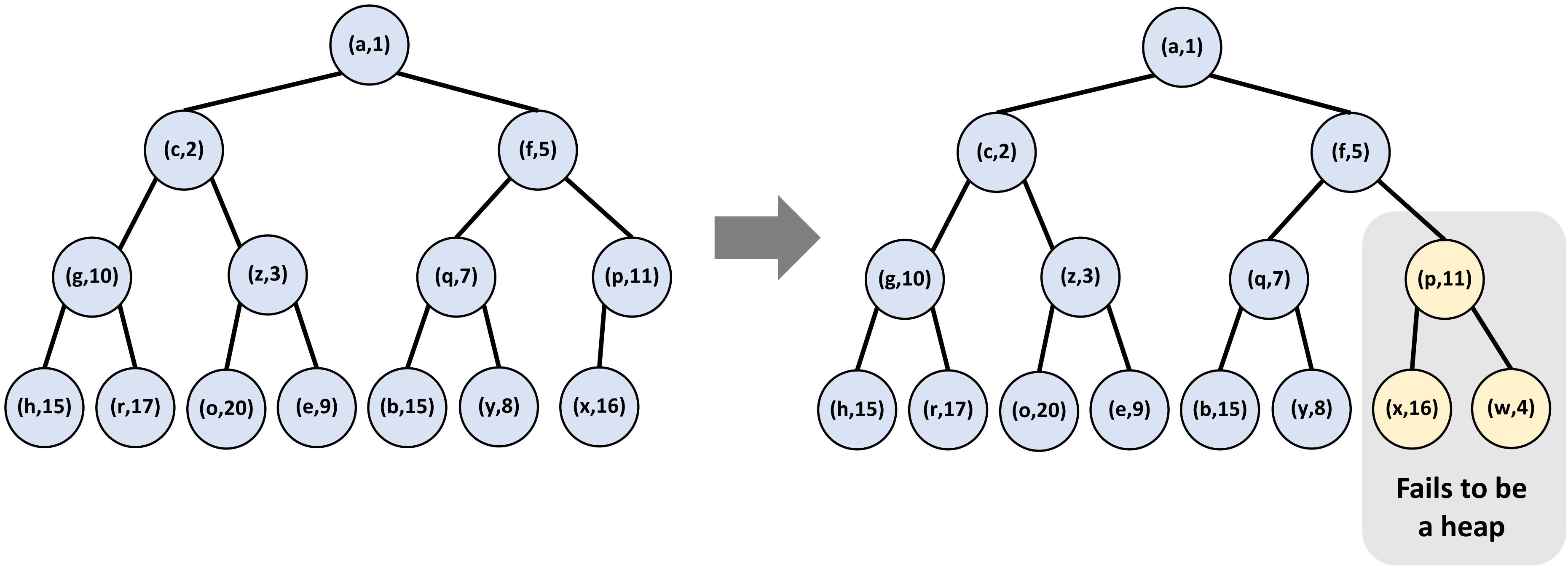


Implementing DecreaseKey

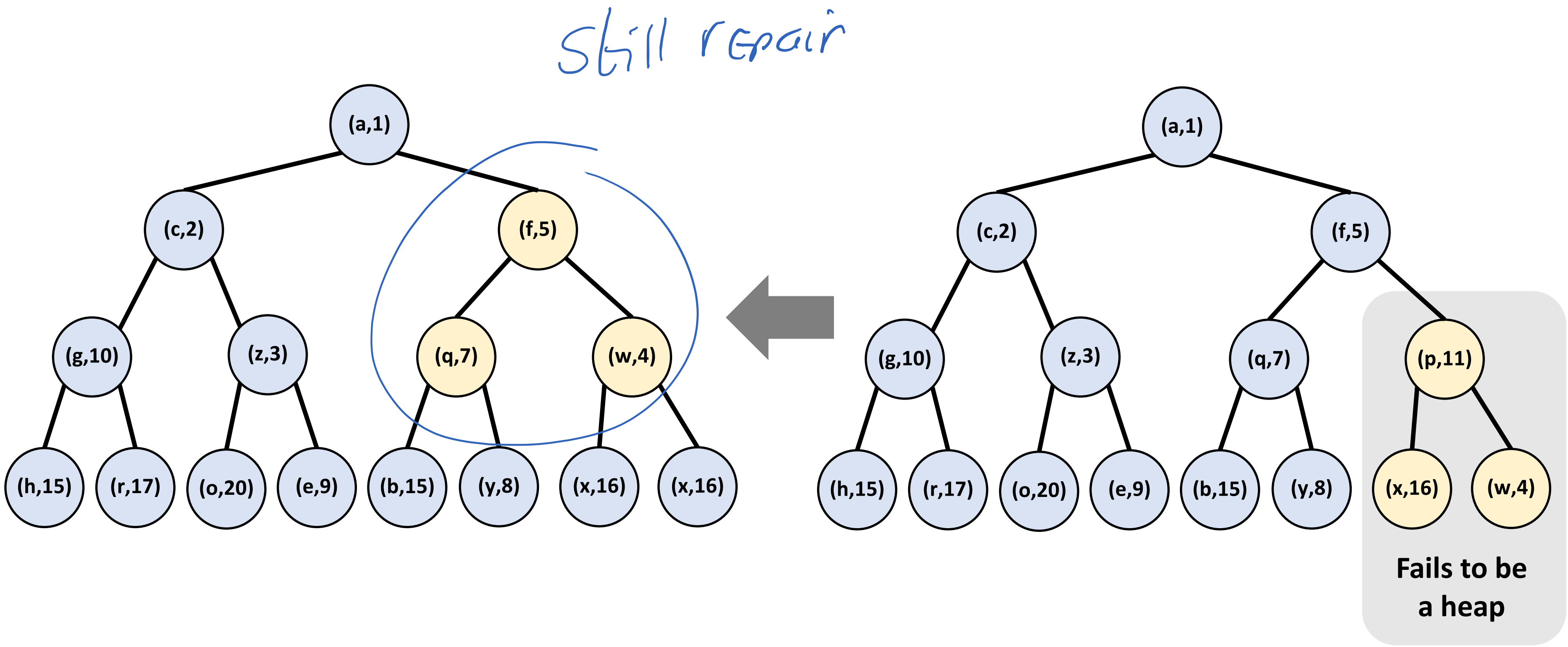
- Two steps:
  - Change the key $O(1)$
  - Repair the heap-order (heapify up) $O(\log n)$

$\log(n)$

Implementing Insert



Implementing Insert

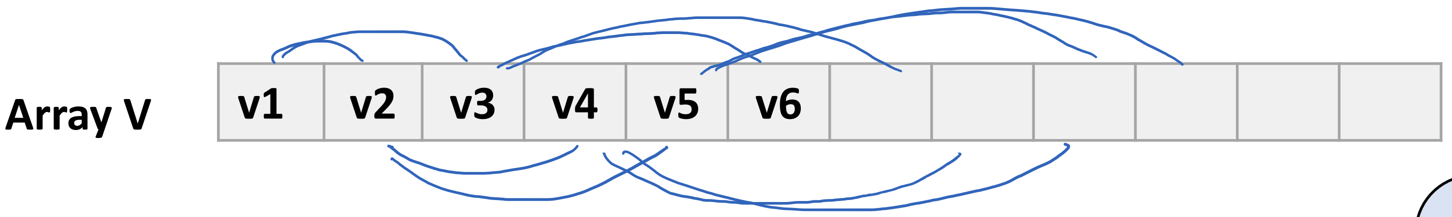


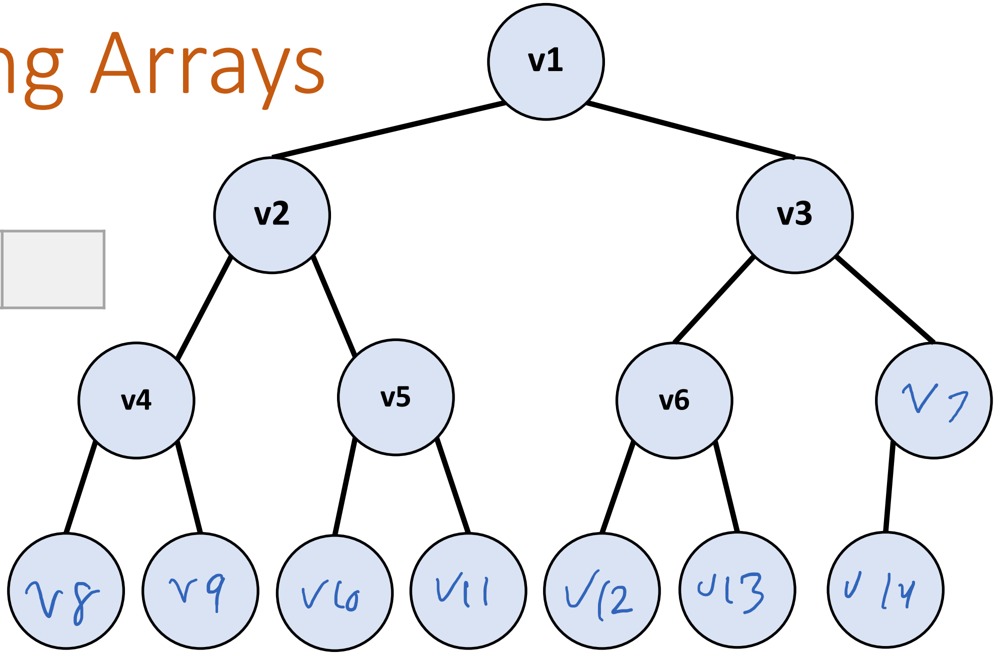
Implementing Insert

- Two steps:
  - Put the new key in the last location $O(1)$
  - Repair the heap-order (heapify up) $O(\log n)$

$O(\log n)$

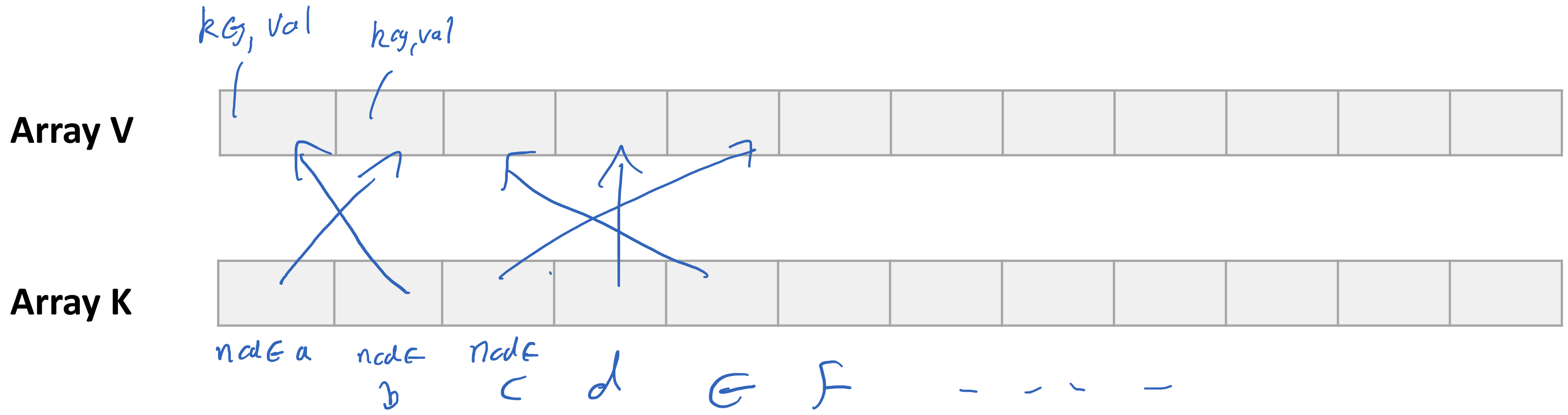
Implementation of Binary Tree Using Arrays





- Maintain an array V holding the values (in order of top to bottom, followed by left to right)
- For any node i in the binary tree, what is the index of
  - LeftChild(i) = $2i$
  - RightChild(i) = $2i+1$
  - Parent(i) = $\lfloor i/2 \rfloor$
- Draw the tree on the array above.

Implementation of Priority Queue Using Arrays



- Maintain an array V holding the (key,value) at each node the binary tree
- Maintain an array K mapping keys index
  - Can find the value for a given key in $O(1)$ time

Binary Heaps

- **Heapify:**
  - O(1) time to fix a single triple
  - With n keys, might have to fix O(log n) triples
  - Total time to heapify is O(log n)

- **Lookup** takes O(1) time
- **ExtractMin** takes O(log n) time
- **DecreaseKey** takes O(log n) time
- **Insert** takes O(log n) time

Implementing Dijkstra with Heaps

Lookup takes O(1) time
ExtractMin takes O(log n) time
DecreaseKey takes O(log n) time
Insert takes O(log n) time

```
Dijkstra(G = (V,E,{ℓ(e)}, s):
  Let Q be a new heap
  Let parent[u] ← ⊥ for every u
  Insert(Q,s,0), Insert(Q,u,∞) for every u != s

  While (Q is not empty):
    (u,d[u]) ← ExtractMin(Q)

    For ((u,v) in E):
      d[v] ← Lookup(Q,v)
      If (d[v] > d[u] + ℓ(u,v)):
        DecreaseKey(Q,v,d[u] + ℓ(u,v))
        parent[v] ← u

  Return (d, parent)
```

build heap

remove one item and heapify

lookup and modification

How much time does Dijkstra take?

n items

n log n

cost per item

Repeat n times

log n

Repeated deg(v) times

log n · deg(v)

Total time: $\sum_v O(\log n) + O(\deg(v)\log n) = O((m+n)\log n) = O(m \log n)$

edges

Dijkstra Summary:

- **Dijkstra's Algorithm** solves **single-source shortest paths** in non-negatively weighted graphs
  - Algorithm can fail if edge weights are negative!

- **Implementation:**
  - A **priority queue** supports all necessary operations
  - Implement priority queues using **binary heaps**
  - Overall running time of Dijkstra: $O(m \log n)$

only paying
log n cost
due to weights

- **Compare to BFS**