Day 5 1/28/2015

RIP & NSP & Spark

A has RIP of order k if $\forall \|x\|_0 \le k$ $\quad (1-\delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta_k)\|x\|_2^2$

A has NSP of order k if $\forall h \in N(A)$ $\quad \|h_S\|_2 \le c \dfrac{\|h_{S^c}\|_1}{\sqrt{k}}$ $\quad \forall\ |S| \le k$

A has spark $\ge k+1$ if $\forall h \in N(A)$ $\quad \|h\|_0 \ge k+1$

Relations: RIP $\Rightarrow$ NSP

RIP is like a well conditioned version of spark.

Spark$(A) \ge k+1 \Rightarrow \forall \|x\|_0 \le k \quad 0 < \|Ax\|_2$ "not in null space"

RIP & Stable Embedding

A is $m \times n$ w/ $m \ll n$

A has RIP $\|x\|_2 \approx \|Ax\|_2$ for sparse x

$A : \mathbb{R}^n \to \mathbb{R}^m$.

So A preserves the magnitude of sparse vectors.
As A linear, preserves the distance between sparse points.

Dimension reduction!

Activity: $A_{m\times n}$ has RIP of order k if $\exists\ \delta_k$ s.t.

$$(1-\delta_k)\|x\|_2 \le \|Ax\|_2 \le (1+\delta_k)\|x\|_2 \qquad \forall\ \|x\|_0 \le k.$$

Do you think the following matrix satisfies RIP of order m?

$$A = \begin{pmatrix} 1 & 1^2 & & 1^n \\ 2 & 2^2 & & 2^n \\ 3 & 3^2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ m & m^2 & & m^n \end{pmatrix}$$

but normalize each column to be length 1.

Is it a good sensing matrix?

Noisy recovery thm

Let A satisfy RIP $(2k)$ with $\delta_{2k} < \sqrt{2}-1$

Let $y = Ax_0 + e$, $\|e\| \leq \varepsilon$.

Then $\hat{x} = \arg\min \|x\|_1$ st $\|Ax - \underbrace{\cancel{Ax_0}}_{y}\| \leq \varepsilon$

satisfies $\|\hat{x} - x\|_2 \leq C_0 \frac{\|x_0 - x_{0,k}\|_1}{\sqrt{k}} + C_2 \varepsilon$.

Proof: Outline (in case $x_{0,k} = 0$)

Technical result: Fix any h. Let Δ be top $2k$ entries

$$\|h_\Delta\|_2 \leq B \frac{|\langle Ah_\Delta, Ah\rangle|}{\|h_\Delta\|_2}$$

optimizer satisfy:

$$\|\hat{x} - x\|_2 \leq C_1 \frac{|\langle Ah_\Delta, Ah\rangle|}{\|h_\Delta\|_2} \quad \text{w/ } h = \hat{x} - x$$

$$\leq C_1 (1+\delta_{2k}) \frac{\cancel{\|h_\Delta\|_2}}{\cancel{\|h_\Delta\|_2}} \|Ah\|$$

As $\|A\hat{x} - y\| \leq \varepsilon$ & $\|Ax_0 - y\| \leq \varepsilon$, $\|A\hat{x} - Ax_0\| \leq 2\varepsilon$

So $\|\hat{x} - x\|_2 \leq C_1 (1+\delta_{2k})\, 2\varepsilon$.

Activity:

A signal is compressible if

$$|x_0|_{(k)} \leq C_r \cdot k^{-r}$$

a) Compute worst case $\|x_0 - x_{0,S}\|_2$

b) Compute worst case

$$\frac{\|x_0 - x_{0,S}\|_1}{\sqrt{S}}$$

c) Could anyone improve bound $\|\hat{x} - x_0\|_2 \leq C \frac{\|x_0 - x_{0,S}\|_1}{\sqrt{S}}$?

d) Isn't every finite signal compressible ??

How good is error estimate?

$$\|\hat{x} - x\|_2 \leq C_2 \varepsilon \qquad \text{for } y = Ax + e \text{ w } \|e\|_2 \leq \varepsilon.$$

Suppose we know which k coefs of x were nonzero, S.

$$y = A_S x$$

restriction of A to columns in S.
overdetermined (tall) matrix $m \times k$

Solve for x by least squares

$$\hat{x} = (A_S^t A_S)^{-1} A_S y.$$

How much error

$$\hat{x} - x_0 = (A_S^t A_S)^{-1} A_S (x_0 + e) - x_0$$

$$\hat{x} - x_0 = (A_S^t A_S)^{-1} A_S \, e$$

$$\|\hat{x} - x_0\| \leq \underbrace{\|(A_S^t A_S)^{-1} A_S\|}_{\substack{\text{like inverse of } A \\ \text{its max sing value} \\ \text{is like reciprocal of} \\ A_S\text{'s min sing value}}} \|e\|_2$$

$$\leq \frac{1}{\sqrt{1 - \delta_{2k}}} \|e\|_2$$

So error estimate optimal up to a constant factor