Activity:

a) Suppose $X \in \mathbb{R}^{2\times 2}$
$X \succeq 0$
$\langle X, e_1 e_1^t \rangle = 0$

What more can you say about X?

b) What if $X \in \mathbb{R}^{n\times n}$?

c) What if $X \in \mathbb{R}^{n\times n}$
$X \succeq 0$
$\langle X, aa^t \rangle = 0$

What more can you say?

Convex Duality under inequality constraints

$\min f(x)$ st $\begin{array}{l} Ax=b \\ x \geq 0 \end{array}$ for $x \in \mathbb{R}^n$

Let $\mathcal{L}(x, \lambda, \nu) = f(x) - \langle \lambda, Ax-b \rangle - \langle \nu, x \rangle$

Let $g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu)$

Dual problem: $\sup_{\substack{\nu \geq 0 \\ \lambda}} g(\lambda, \nu)$

Weak duality: Let p^* be primal optimal value, d^* be dual optimal value. $p^* \geq d^*$

Strong duality: $p^* = d^*$ and d^* is achieved

Complementary slackness:

If problem has primal & dual optimal values that are achieved, and equal, then

$$\langle \nu^*, x^* \rangle = 0$$

That is, $\forall i$ $\nu_i^* > 0 \Rightarrow x^* = 0$ & $x_i^* > 0 \Rightarrow \nu_i^* = 0$

Roughly: i^{th} lagrange multiplier is 0 unless constraint is active

KKT conditions

If f is smooth & convex, a sufficient condition for x^* & (λ^*, ν^*) to be primal & dual optimal are

- primal feasibility, dual feasibility, complementary slackness, and stationarity

$$A x^* = b \quad \} \text{ primal feasibility}$$
$$x^* \geq 0$$
$$\nu^* \geq 0 \quad \} \text{ dual feasibility}$$
$$\langle \nu^*, x^* \rangle = 0 \quad \} \text{ complementary slackness}$$
$$0 = \nabla f(x^*) - A^t \lambda^* - \nu \quad \} \text{ stationarity}$$

This is to say,
(λ^*, ν^*) certifies optimality of x^*.

Derivation of KKT conditions for PhaseLift

$$\min \quad tr(X) \quad st \quad \mathcal{A}X = b, \quad X \succeq 0$$

where $\mathcal{A} : S_n \to \mathbb{R}^m$, $X \mapsto \{a_i^t X a_i\}_{i=1 \ldots m}$ $\qquad$ $\mathcal{A}^* : \mathbb{R}^m \to S_n$, $\lambda \mapsto \sum_i a_i a_i^t$

$$\mathcal{L} = \langle I, X \rangle - \langle \lambda, \mathcal{A}X - b \rangle - \langle Q, X \rangle$$

$$0 = \nabla \mathcal{L} \Rightarrow I - \mathcal{A}^*\lambda - Q = 0$$

Dual feasibility: $Q \succeq 0$

Complementary slackness: $\langle Q, X_0 \rangle = 0$

KKT conditions for X_0 to be minimizer

$$Q = I - \mathcal{A}^*\lambda$$
$$Q \succeq 0$$
$$\langle Q, X_0 \rangle = 0$$

If $X_0 = e_1$, let $T = \{$sym matrices supported in first row/col$\}$

$T^\perp = \{$sym matrices supported in lower right $n-1 \times n-1$ block$\}$

KKT conditions become (at $X_0 = e_1 e_1^t$)

$$Y = \mathcal{A}^*\lambda$$
$$Y_T = e_1 e_1^t$$
$$Y_{T^\perp} \preceq I_{T^\perp}$$

Derivation of KKT conditions for PhaseLift feasibility

$$\min \quad 0 \quad \text{st} \quad \begin{aligned} AX &= b \\ X &\succeq 0 \end{aligned}$$

$$\mathcal{L} = -\langle \lambda, AX-b \rangle - \langle Q, X \rangle$$

$$0 = \nabla \mathcal{L} \Rightarrow \quad -A^*\lambda - Q = 0$$

Dual feasibility: $Q \succeq 0$

Complementary slackness: $\langle Q, X_0 \rangle = 0$

KKT conditions

$$Q = -A^*\lambda$$
$$Q \succeq 0$$
$$\langle Q, X_0 \rangle = 0$$

If $X_0 = e_1 e_1^t$, KKT conditions become

$$Y = A^*\lambda$$
$$Y_T = 0$$
$$Y_{T^\perp} \preceq 0$$

Recovery by exact dual certificate

Let $x_0 = e_1$. Let $X_0 = e_1 e_1^t$. Let $b = A\bar{X}_0$. (*)

Find X such that $X \succeq 0$, $AX = b$.

Lemma: If $\exists\, Y = A^*\lambda$ such that $Y_T = 0$ and $Y_{T^\perp} \prec 0$ and A is injective on T, then $\bar{X}_0$ is ! soln to (*)

Proof:

Suppose $X_0 + H \succeq 0$ and $A(X_0 + H) = b$.

So $AH = 0$.

$$\Rightarrow \quad 0 = \langle \lambda, AH \rangle = \langle A^*\lambda, H \rangle$$
$$= \langle Y, H \rangle$$
$$= \cancel{\langle Y_T, H_T \rangle}^{\,0} + \langle Y_{T^\perp}, H_{T^\perp} \rangle$$
$$= \langle Y_{T^\perp}, H_{T^\perp} \rangle$$

$\Rightarrow \quad H_{T^\perp} = 0$ b/c $Y_{T^\perp} \prec 0$

So H lives on T. But A^* injective on T, so $H = 0$ ∎

Visually, Y defines a hyperplane that separates feasible points from ~~interior~~ SDP cone.