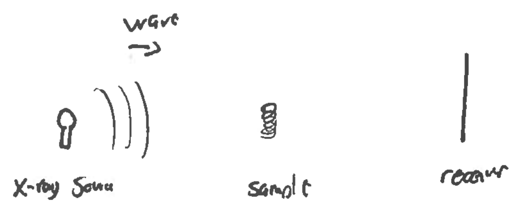
Phase Retrieval problem:

Let $x_0 \in \mathbb{R}^n$ unknown. Let $a_i \in \mathbb{R}^n$ known $i = 1 \cdots m$

Given $|\langle a_i, x_0 \rangle|$, find x_0.

Application: X-ray crystallography



At receiver, measure squared modulus of diffraction pattern

Why the name?

Measurements $|\langle a_i, x_0 \rangle|$ lose $\pm$ sign (or in complex case, the phase $e^{\cancel{2\pi} i\theta}$. If we had these phases, we would know $\langle a_i, x_0 \rangle$ and could solve for x_0 by linear algebra.

Questions:

How many measurements do we need?
Algorithm for recovery
Robustness to noise and gross errors?

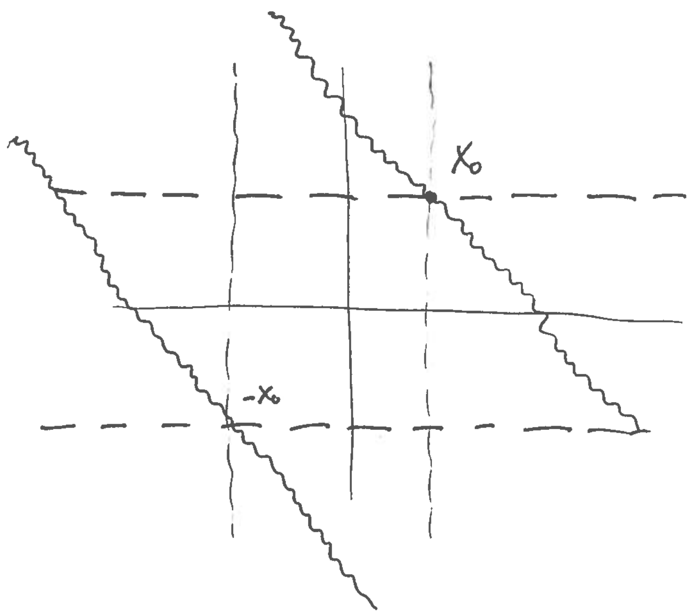
Degeneracies: x_0 & $-x_0$ give same measurements in $\mathbb{R}$ case

x_0 & $e^{i\theta} x_0$ — — ——— — $\mathbb{C}$ case

Can only recover x_0 up to global phase

Activity:

Draw a picture of recovery task in $\mathbb{R}^2$ with 3 measurements

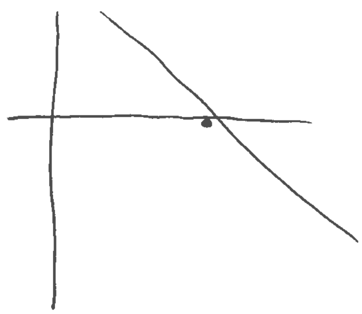


Find points where these pairs of lines intersect.

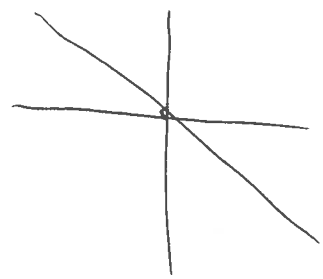
Combinatorial Algorithm

Let $b_i := |\langle a_i, x_0 \rangle|$.

Loop over all signs $\langle a_i, x_0 \rangle = \pm b_i$
Search for consistency



Not consistent



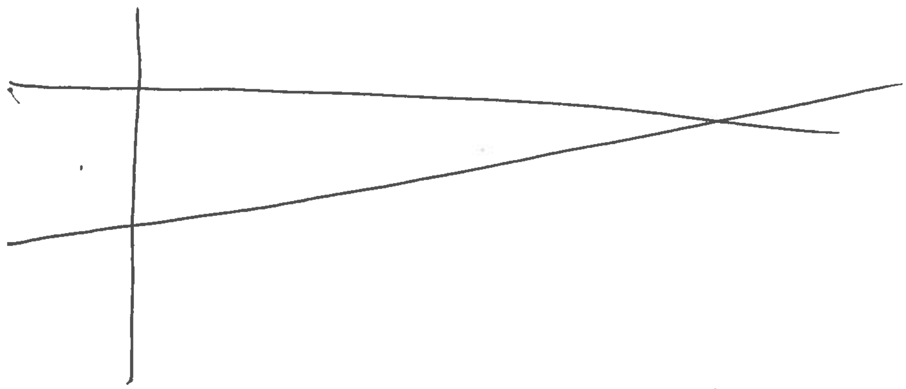
Consistent

Problems:

- exponential work
- in complex case would have to try continuum of cases

Activity:

Draw the point that minimizes sum of squares of distances to these lines



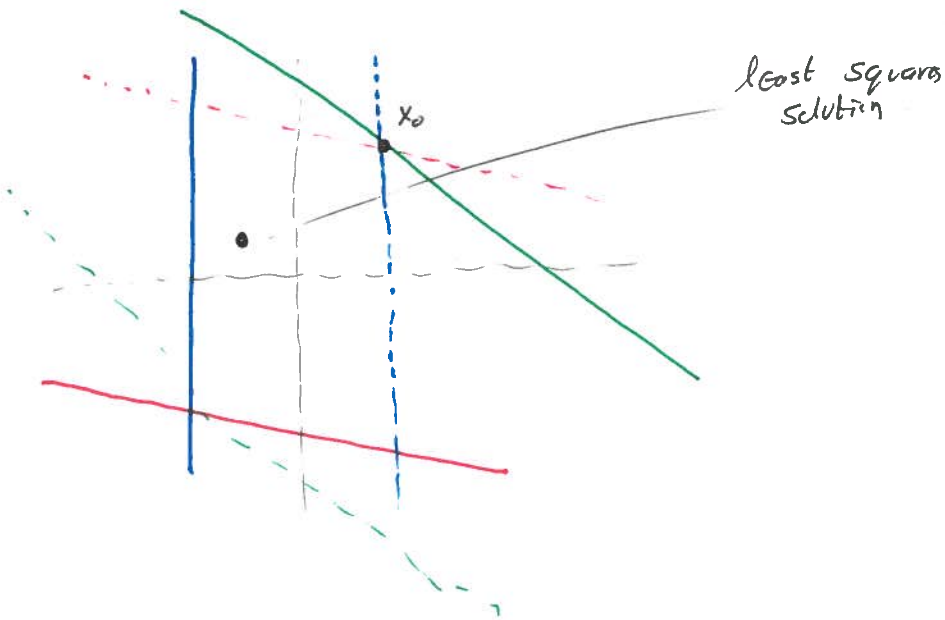
Each line is eqn $a_i \cdot x = b_i$ for $\|a_i\| = 1$

$$\begin{pmatrix} -a_1- \\ -a_2- \\ -a_3- \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

What is $\underset{x}{\operatorname{argmin}} \|Ax-b\|_2$

Greedy Algorithm

- Initialize signs of $\langle x_0, a_i \rangle$
- Solve $\langle x, a_i \rangle = \pm b_i$
- update signs by which branch is closer



can get stuck