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Signal Recovery
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HW 2 [Revised]

Due: 21 February 2017 in class

1. Let $Z \sim \mathcal{N}(0,1)$. Are the following random variables subgaussian? Are they subexponential? Prove your answer.
   (a) Z^2
   (b) Z^4
   (c) $Z^4 1_{|Z|\leq 4}$ (Here 1_E is the indicator of the event E)
   (d) Geometric distribution with probability p
2. Let A be an $N \times n$ real valued matrix. Show that the minimum singular value of A is Lipschitz continuous with Lipschitz constant 1 as a function of A (with respect to the ℓ^2 norm if A is considered as an Nn dimensional vector).
3. Let $x_1, x_2, x_3 \sim \mathcal{N}(0, I_n)$ be independent. The goal of this problem is to argue that the triangle formed by these three points is close to being equilateral.
   (a) By using a union bound, state and prove a high probability concentration result that the lengths of the three edges are all within a factor of 1.01 of each other. Your bound should have a probability that approaches 1 as n approaches ∞.
   (b) By using a union bound, state and prove a high probability concentration result that each of the triangle's interior angles are between $0.99 \cdot \pi/3$ and $1.01 \cdot \pi/3$. Your bound should have a probability that approaches 1 as n approaches ∞.
4. (Revised) Let $a_1, \dots a_N$ be i.i.d. $\mathcal{N}(0, I_n)$ vectors. Let $\sigma_i \sim \begin{cases} 1 & \text{with prob. } 1/2 \\ -1 & \text{with prob. } 1/2, \end{cases}$ be independent from a_i and each other. Let $A = \sum_{i=1}^{N} \sigma_i a_i a_i^\intercal$.
   (a) Prove that there exists constants c, C such that with probability at least $1 - 2e^{-cn}$,
   $$\|A\| \leq C(\sqrt{Nn} + n).$$
   Here, $\|A\|$ is the spectral norm of A.
   (b) Provide a qualitative explanation of the form of this bound on the spectral norm. As part of your answer: why would be bound be false without the $\sqrt{Nn}$ term? Why would the bound be false without the n term? Can you provide an intuitive explanation of why the $\sqrt{Nn}$ has the structure it does? That is, if you hadn't been told this upper bound on the spectral norm of A, how could you have reasonably guessed it?